Last time

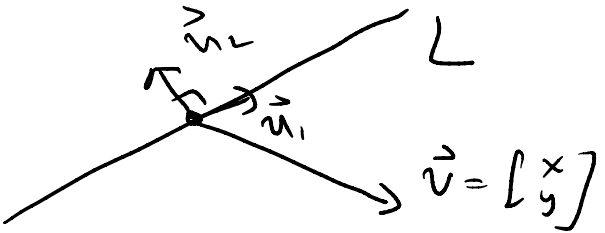
- Coordinates

The coordinates of the vector $\vec{x}$ in the basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_r\}$ are the scalars $c_1, \dots, c_r$ such that $\vec{x} = c_1\vec{v}_1 + \dots + c_r\vec{v}_r$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}.$$

Ex $\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$



$\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ a basis for $\mathbb{R}^2$

$$S = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \qquad S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

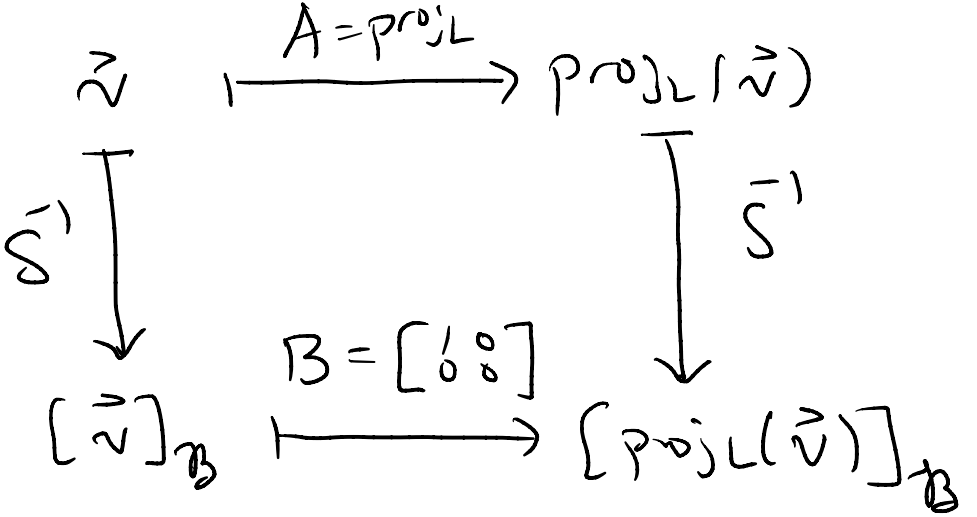
$$[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3x+4y \\ -4x+3y \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1 \cdot \vec{v} \\ \vec{u}_2 \cdot \vec{v} \end{bmatrix}.$$

On the other hand, $\text{proj}_L(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1$,

so
$$[\text{proj}_L(\vec{v})]_{\mathcal{B}} = [(\vec{u}_1 \cdot \vec{v})\vec{u}_1]_{\mathcal{B}}$$
$$= (\vec{u}_1 \cdot \vec{v})[\vec{u}_1]_{\mathcal{B}}$$
$$= (\vec{u}_1 \cdot \vec{v})\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \vec{u}_1 \cdot \vec{v} \\ 0 \end{bmatrix}.$$

Upshot Projection is simplified by working in coordinates adapted to it.



The matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ represents the linear transformation $proj_L$ in the basis $\mathcal{B}$. The relationship between the two is

$$\boxed{A = SBS^{-1}}$$

Def Two $n \times n$ matrices A and B are similar if there is an invertible matrix S such that

$$A = SBS^{-1}$$

Similar matrices represent the same linear transformation in different bases.

Def Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ be a basis for $\mathbb{R}^m$. The $\mathcal{B}$-matrix of the

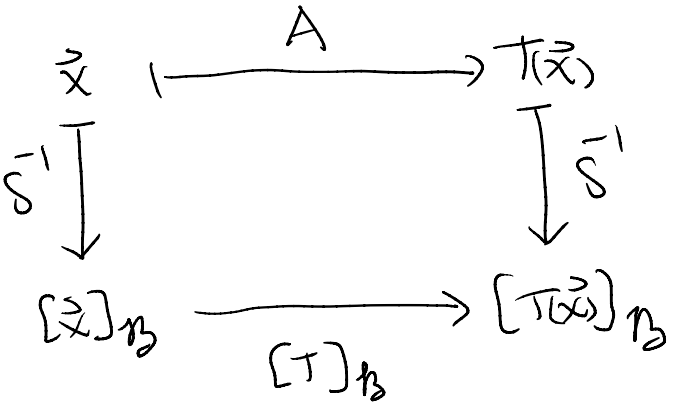
Linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is the matrix $[T]_{\mathcal{B}}$ such that

$$[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}.$$

If A is the matrix of T, then

$$[T]_{\mathcal{B}} = S^{-1}AS.$$

The columns of $[T]_{\mathcal{B}}$ are the vectors $[Tv_i]_{\mathcal{B}}$.



Ex $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

T the linear transformation with matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = S^{-1}AS = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 2 & 6 \end{bmatrix}.$$

The coordinates in the example led to a simpler situation in that the matrix of $proj_L$ in the basis $\mathcal{B}$ was diagonal. This happens precisely when the value of the linear transformation on each basis vector $\vec{v}_i$ is parallel to $\vec{v}_i$. Such good coordinates might or might not exist.