HOMEWORK 2: TOPOLOGY 1, SPRING 2022
DUE MARCH 9

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. The word *show* is synonymous with *prove*. This assignment has five problems and two pages.

(1) *Compact and connected quotients.* Let $p : X \to Y$ be a quotient map.
   (a) Show that, if Y and $p^{-1}(y)$ are connected for every $y \in Y$, then X is connected (Munkres 3.23.11).
   (b) Assume further that p is closed. Show that, if Y and $p^{-1}(y)$ are compact for every $y \in Y$, then X is compact (see Munkres 3.26.12 for a hint/spoiler).

(2) *Compactness and continuity.* Let X and Y be spaces.
   (a) Show that, if Y is compact, then the projection $X \times Y \to X$ is closed (Munkres 3.26.7).
   (b) Show that, if Y is compact and Hausdorff, then $f : X \to Y$ is continuous if and only if the *graph* of f, which is the subspace

$$G_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y,$$

   is a closed subspace (see Munkres 3.26.8 for a hint/spoiler).

(3) *Stars.* A subset $A \subseteq \mathbb{R}^n$ is *star-shaped* (or *star convex*) if there is a point $a_0 \in A$ such that, for every $a \in A$, the line segment joining a_0 and a lies in A.
   (a) Are convex sets star-shaped? Are star-shaped sets convex? For each question, give either a proof or a counterexample.
   (b) Show that, if A is star-shaped, then A is simply connected.
   This problem is mostly Munkres 9.52.1.

(4) *Retractions.* Let $i : A \to X$ be the inclusion of a subspace. Recall that a map $r : X \to A$ is called a retraction of i if $r \circ i = \mathrm{id}_A$, in which case A is called a retract of X.
   (a) Show that a retract of a path connected space is path connected.
   (b) Show that, if r is a retraction of i, then i induces an injection and r induces a surjection at the level of fundamental groups.
   (c) Show that a retract of a simply connected space is simply connected.
   (d) Show that the circle is not a retract of the unit disk (you may take $\pi_1(S^1)$ as known, even if we haven't finished calculating it in class).
   This problem is partly Munkres 9.52.4. There are spoilers in Section 9.55.

(5) *Contractibility.* A map is said to be *nullhomotopic* if it is homotopic to a constant map. A space is said to be *contractible* if its identity map is nullhomotopic.
   (a) Show that X is contractible if and only if X is homotopy equivalent to a topological space with one point.
   (b) Show that a retract of a contractible space is contractible.
   (c) Show that convex and star-shaped subsets of $\mathbb{R}^n$ are contractible.

(d) Show that, if Y is contractible, then any two maps from X to Y are homotopic. Deduce that contractible spaces are path connected.
(e) Show that, if X is contractible and Y is path connected, then any two maps from X to Y are homotopic.

This problem is partly Munkres 9.51.3, 9.58.5, and 9.58.6. You might find 9.51.2 helpful as a hint/spoiler.