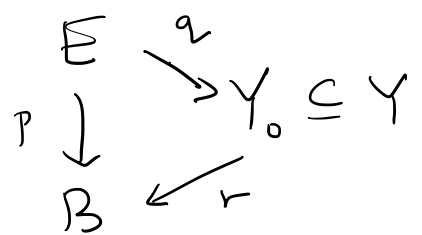
Last time

- Existence of the universal cover

Before continuing, we pause to justify the adjective "universal".

Thm Let $p: E \to B$ and $r: Y \to B$ be covering maps with E simply connected. For every path component $Y_0 \subseteq Y$, there is a covering map $q: E \to Y_0$ such that $p = r|_{Y_0} \circ q$



Lemma 1 If $p: E \to B$ is a covering map, then so is $p|_{E_0}$ for every path component $E_0 \subseteq E$.

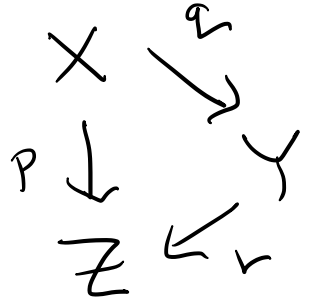
Proof We claim that $p|_{E_0}$ is surjective, for which it suffices to show that $p(E_0)$ is closed. Indeed, E is locally path connected, since B is, so E_0 is open; therefore, since covering maps are open (exercise), $p(E_0)$ is open. If $p(E_0)$ is also closed, then $p(E_0)$ and $B \setminus p(E_0)$ are a separation, but B is connected, since path connected.

Choose $x \in \overline{p(E_0)}$ and a path connected open $x \in U$ evenly covered by p. Since $U \cap p(E_0) \neq \emptyset$, also $p^{-1}(U) \cap E_0 \neq \emptyset$, so some slice of $p^{-1}(U)$ intersects E_0, hence is contained therein,

since E_0 is a path component and U is path connected. Therefore, $U \subseteq p(E_0)$, so $x \in p(E_0)$, as desired.

Consideration of the previous argument shows that $p|_{E_0}$ is a covering map (exercise). □

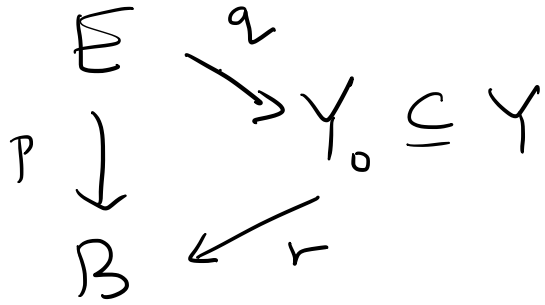
Lemma 2 In the commuting diagram



of path connected and locally path connected spaces, if p & r are covering maps, then so is q.

Proof Exercise, or see Lemma 80.2(a) in Munkres. □

Thm Let $p: E \to B$ and $r: Y \to B$ by covering maps with E simply connected. For every path component $Y_0 \subseteq Y$, there is a covering map $q: E \to Y_0$ such that $p = r|_{Y_0} \circ q$



Proof By Lemma 1, $r|_{Y_0}$ is a covering map. Since E is path connected and locally path connected and $p_*(\pi_1(E)) = \{1\} \subseteq r_*(\pi_1(Y_0))$, there is a unique lift q of p along r, which is a covering map by Lemma 2. □

So any covering map with a path connected total space factors through $p: E \to B$ if E is simply connected, justifying the adjective "universal".

Running assumptions B is path connected, locally path connected, and semilocally simply connected.

For such B, the image of the function

$$\left\{\begin{array}{l}\text{path connected}\\ \text{covering spaces}\\ \text{w/ basepoint}\end{array}\right\} \Big/ \text{equivalence preserving basepoint} \longrightarrow \left\{\begin{array}{l}\text{subgroups}\\ \text{of } \pi_1(B, b_0)\end{array}\right\}$$

contains the trivial subgroup. What about others?

Recall The lifting correspondence induces a bijection

$$\pi_1(B, b_0)/H \xrightarrow{\cong} \bar{p}^{-1}(b_0)$$

depending on a choice of $e_0 \in \bar{p}^{-1}(b_0)$.

Recall By the orbit stabilizer theorem, if X is a transitive right G-set, choosing $x \in X$ gives

$$G/H \xrightarrow{\cong} X$$
$$Hg \longmapsto xg,$$

where $H := G_x$ is the stabilizer of x. Thus,

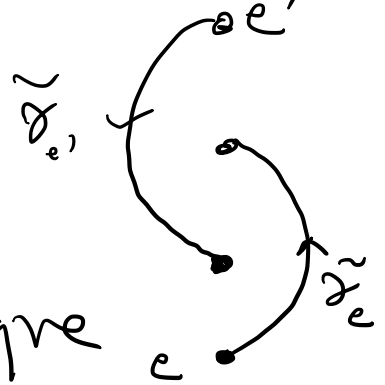
$$\left\{ \begin{array}{l} \text{transitive } G\text{-sets} \\ \text{w/ choice of element} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subgroups} \\ \text{of } G \end{array} \right\}$$

Set $G := \pi_1(B, b_0)$, and define

$$\bar{p}^{-1}(b_0) \times G \longrightarrow \bar{p}^{-1}(b_0)$$

$$(e, [\gamma]) \longmapsto e \cdot [\gamma] := \tilde{\gamma}_e(1)$$

where $\tilde{\gamma}_e$ is the unique lift of γ with $\tilde{\gamma}(0) = e$.



Lemma the function $(e, [\gamma]) \mapsto e \cdot [\gamma]$ is a G-action.

Proof First, it is well-defined by unique lifting of path homotopies. Second, since the unique lift of the constant loop is constant for every $e \in \bar{p}^{-1}(b_0)$, we have $e \cdot 1 = e$. Finally, note that

$(e \cdot [\gamma]) \cdot [\gamma']$ is the endpoint of the unique lift of γ' starting at $\tilde{\gamma}_e(1)$, while $e \cdot ([\gamma] \cdot [\gamma'])$ is the endpoint of the unique lift of $\gamma * \gamma'$ starting at e, and these coincide. □

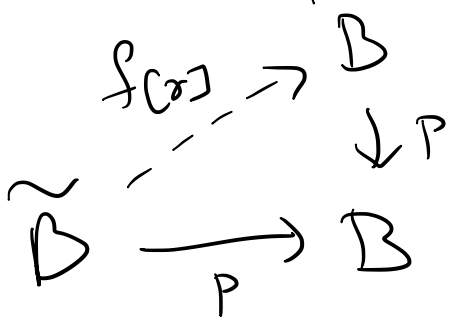
Since the stabilizer of any $e \in \bar{p}^{-1}(b_0)$ is $H := p_*(\pi_1(E, e))$ (exercise), the bijection of the lifting correspondence is simply orbit-stabilizer applied to the G-set $\bar{p}^{-1}(b_0)$. In particular, it is an isomorphism of G-sets.

Idea The fiber of the universal cover is the G-set G, and the fiber of the (hypothetical) covering space corresponding to H is G/H. So construct the latter from the former by "quotienting by H".

Question How does G act on the universal cover?

Prop Let $\tilde{B} \xrightarrow{p} B$ denote the universal cover, and fix $e_0 \in \tilde{p}^{-1}(b_0)$. Given $[\gamma] \in G$, there is a unique equivalence $f_{[\gamma]} : \tilde{B} \to \tilde{B}$ such that $f_{[\gamma]}(e_0) = \tilde{\gamma}_{e_0}(1)$.

Proof Since $p_*(\pi_1(E, e_0)) = p_*(\pi_1(E, \tilde{\alpha}_{e_0}(1))) = \{1\}$, the lifting lemma gives the unique lift



Uniqueness implies that $f_{[\alpha]} \circ f_{[\alpha]^{-1}} = f_{[\alpha]} \circ f_{[\alpha]} = id_{\tilde{B}}$, so $f_{[\alpha]}$ is a homeomorphism, hence an equivalence. □

Now, we were looking for a G-action. Certainly, if $[\alpha] = 1$, then $f_{[\alpha]} = id_{\tilde{B}}$. On the other hand, given $[\alpha]$ and $[\beta]$, by construction we have that

$$f_{[\alpha]} f_{[\beta]}(e_0) = f_{[\alpha]}(e) \qquad e := \widetilde{\beta}_{e_0}(1)$$

$$= \widetilde{\beta}_{e_1}(1) \qquad e_1 := \widetilde{\alpha}_{e_0}(1)$$

$$= \widetilde{\alpha}_{e_0} * \widetilde{\beta}_{e_1}(1)$$

$$= \widetilde{(\alpha * \beta)}_{e_0}(1)$$

$$= f_{[\alpha][\beta]}(e_0),$$

hence $f_{[\alpha]} f_{[\beta]} = f_{[\alpha][\beta]}$ by uniqueness.

So we have defined a G-action — but a <u>left</u> G-action! We therefore have

two actions of G on $\tilde{p}^{-1}(b_0)$, one on the left depending on a choice of e_0, and one on the right that is canonical. Under the bijection $G \cong \tilde{p}^{-1}(b_0)$ given by the choice of e_0, these actions correspond to the respective actions of G on itself by left and right multiplication.