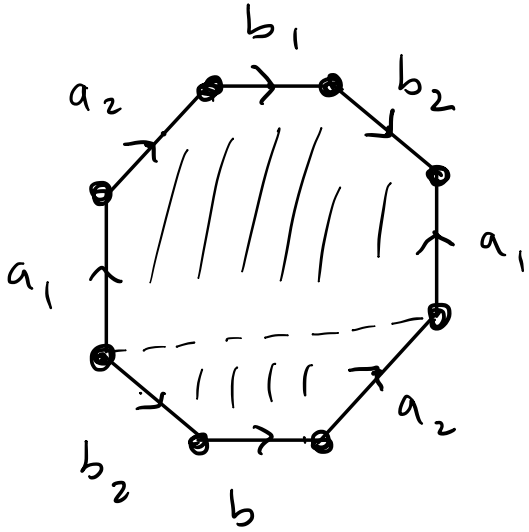
Last time

- Polygonal regions & labeling schemes
- $\pi_1(P(a,\varepsilon)) \cong \langle a_1, \dots, a_n \mid a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \rangle$
- S^2, $\#_n T^2$, $\#_m \mathbb{RP}^2$ distinct
- Proper schemes, projective & torus type

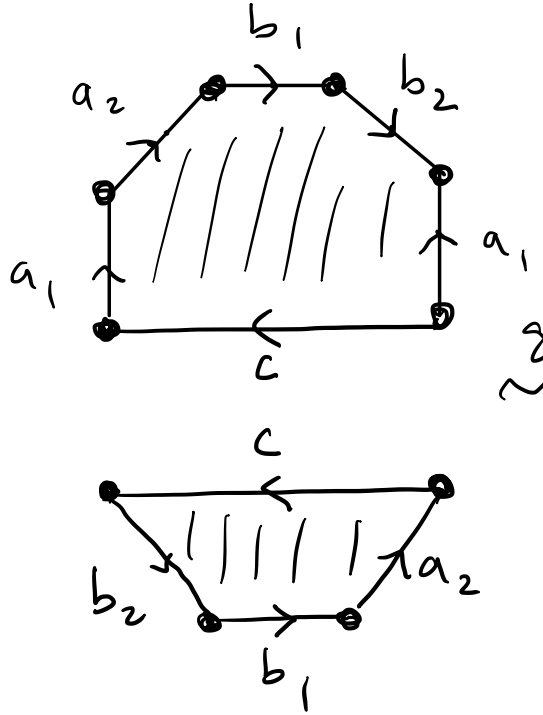
Thm Any surface arising from a proper labeling scheme is homeomorphic to S^2, $\#_n T^2$, or $\#_m \mathbb{RP}^2$.

Last time we indicated how to rewrite a proper scheme in the form $a_1^2 \cdots a_n^2 w$ with w empty or of torus type.

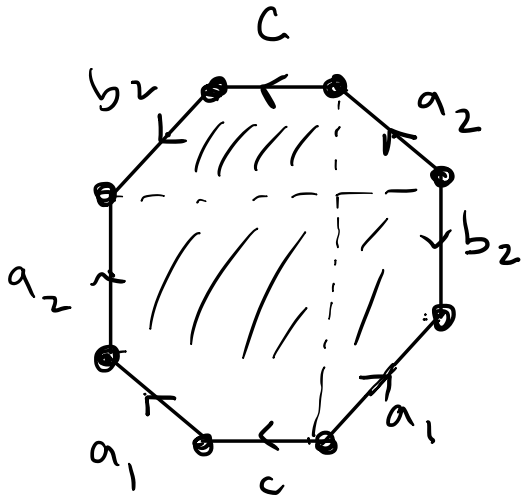
Ex | Consider the scheme $a_1 a_2 b_1 b_2 a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1}$ of torus type



cut

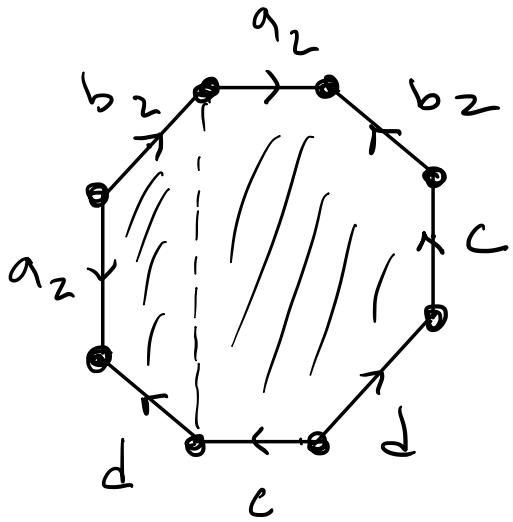


glue



cut

$c^{-1} d^{-1} c d a_2^{-1} b_2 a_2 b_2^{-1} =$



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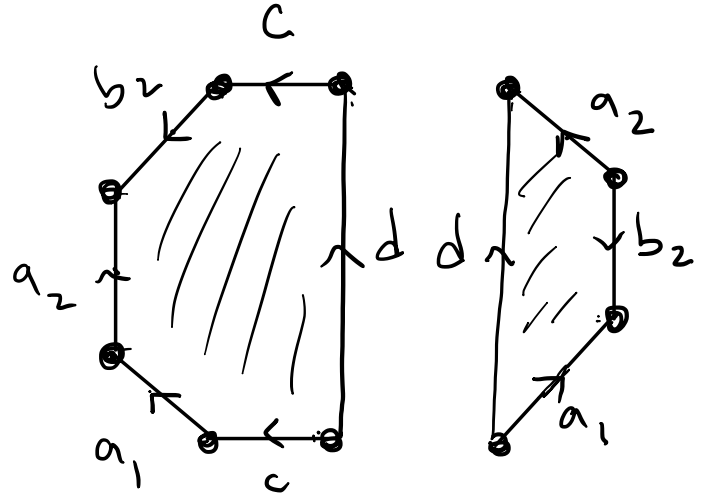
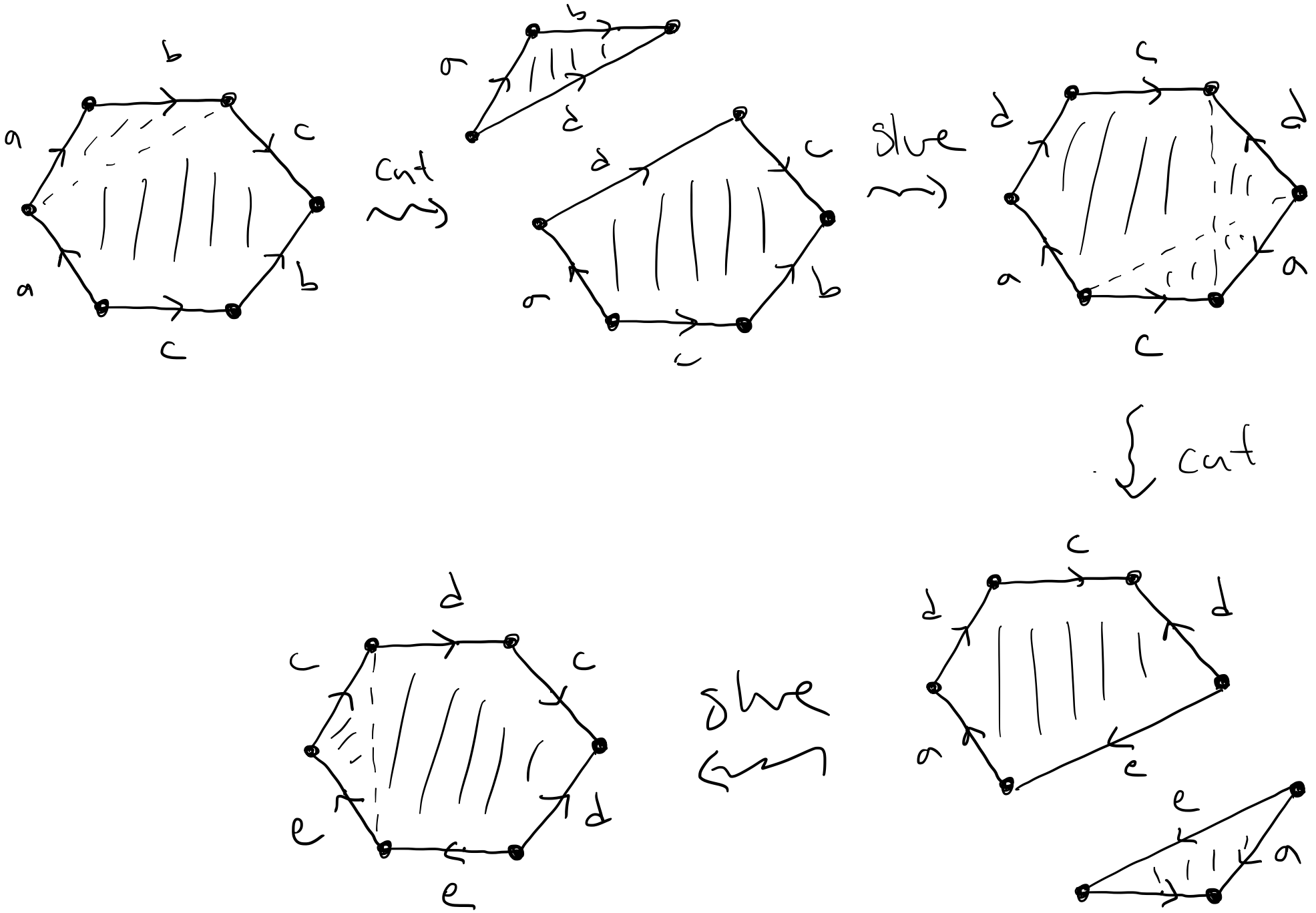


Relabeling, we obtain the labeling scheme for $T^2 \# T^2$.

Applying this idea in greater generality, one shows that it is always possible to pull a word of the form $ab\bar{a}\bar{b}^1$ to the front of a scheme of torus type.

It remains to consider the interaction of projective and torus factors.

Ex 2 Consider the scheme $aabcb^{-1}c^{-1}$, which corresponds to the surface $\mathbb{RP}^2 \# T^2$ obtained by slicing a projective plane to a torus after removing a disk from each.



Relabeling, we obtain $aabcb\bar{c}^1$, and we have already seen how to transform $bcb\bar{c}^1$ into

bbcc. We conclude that $aabcb^{-1}c^{-1}$ and $aabbcc$ determine homeomorphic surfaces, i.e., that $\mathbb{RP}^2 \# T^2 \cong \#_3 \mathbb{RP}^2$.

Sketch proof of thm Begin with a word w representing a proper labeling scheme. If w is of torus type, then, using the technique of Example 1, we may assume that $w = ab a^{-1} b^{-1} w'$, where w' is either empty or of torus type and strictly shorter than w. Induction on the length of w shows that that the resulting surface is homeomorphic to $\#_n T^2$ for some n.

If w is not of torus type, we may assume by the proposition that $w = a_1 a_1 \cdots a_k a_k w'$, where w' is empty or of torus type. If $w' = \phi$, the resulting surface is $\#_n \mathbb{RP}^2$. Otherwise, by the same reasoning as before, we may assume that $w = a_1 a_1 \cdots a_k a_k a b a^{-1} b^{-1} w''$, and the technique of Example 2 shows that the resulting surface is homeomorphic to the surface arising from $a_1 a_1 \cdots a_k a_k a a b b w''$. Induction on the length of w'' yields the claim.

□

Thus, any surface obtainable by "origami" from a polygonal region is homeomorphic to one from our list. We will now sketch the proof that any surface is so obtainable, as long as it admits a triangulation. It is a classical theorem, which we will not discuss, that every surface is triangulable.

Def A <u>triangulation</u> of a surface Σ is a collection of continuous functions $\{h_i : T_i \to \Sigma\}_{i=1}^{r}$ with the following properties.

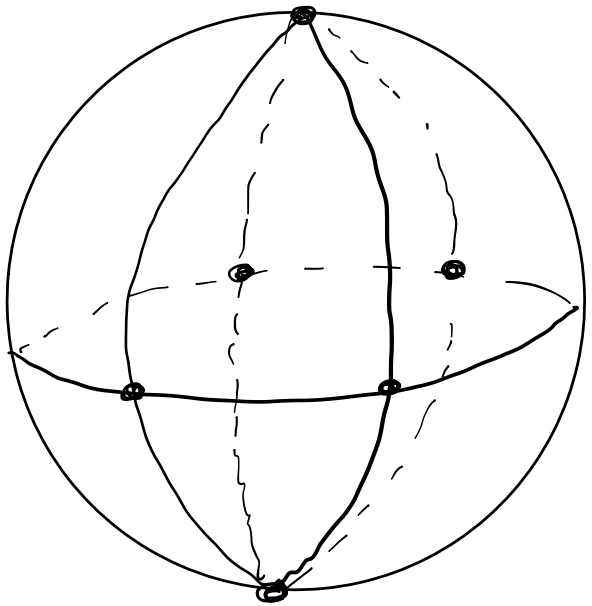
(1) Each T_i is a triangular region in $\mathbb{R}^2$.

(2) Each h_i is a homeomorphism onto its image.

(3) $\Sigma = \bigcup_{i=1}^{n} \mathrm{im}(h_i)$

(4) For each $i \neq j$, $m(h_i) \cap m(h_j)$ is either empty, the image of both a vertex of T_i and a vertex of T_j, or the image of both an edge of T_i and an edge of T_j.

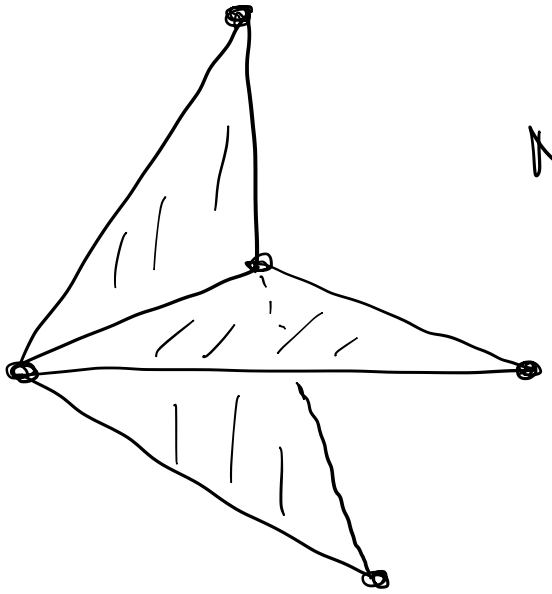
(5) In the latter case, the restriction of $h_j^{-1} h_i$ to the relevant edge of T_i is linear



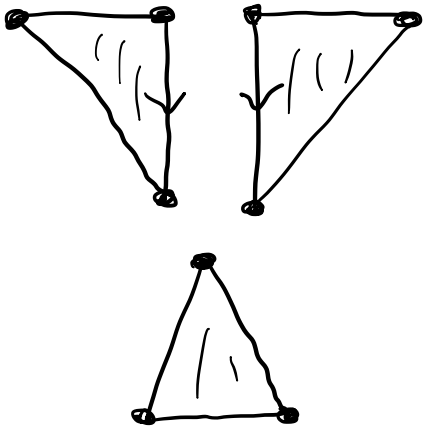
A triangulation of S^2

Thm If Σ admits a triangulation, then Σ is homeomorphic to the quotient space obtained from a disjoint union of triangular regions in $\mathbb{R}^2$ by identifying edges in pairs.

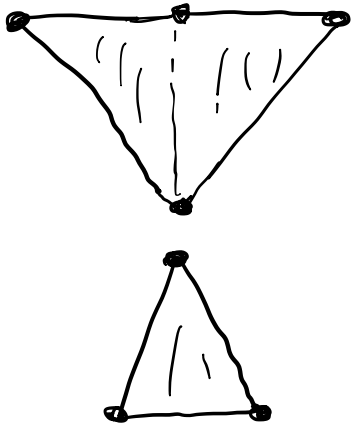
Potential problems:

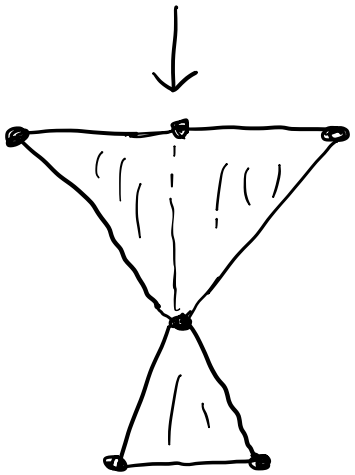


Not in pairs



identify in pairs

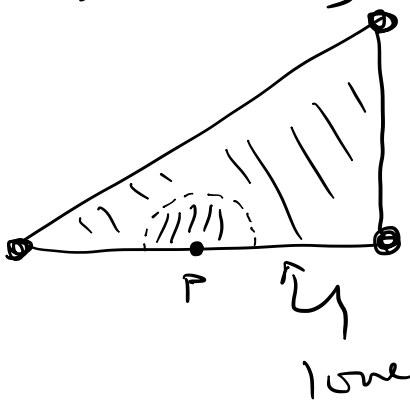




identify further

Proof We may assume that $\bigcap_{i=1}^{r} T_i = \emptyset$, so the h_i induce a map $h: \bigcup_{i=1}^{r} T_i \to \Sigma$, which is a quotient map, since it is closed (compact source and Hausdorff target) and surjective by (3).

First, we claim that every edge is identified with at least one other edge. Otherwise, a point p on the edge in question has arbitrarily small neighborhoods homeomorphic to the upper half-disk $\mathring{D}^2 \cap \{y \geq 0\}$, hence arbitrarily small open neighborhoods U such that $U \setminus \{p\}$ is simply connected.



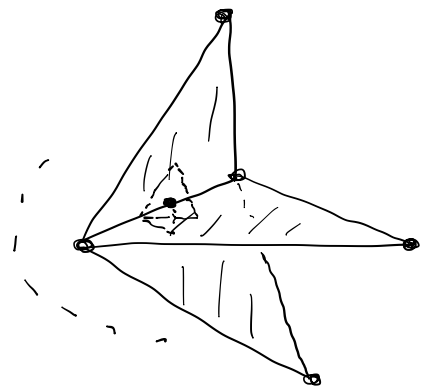
Since Σ is a surface, p also

has arbitrarily small open neighborhoods V homeomorphic to $\mathbb{R}^2$. Then the composite

$$\mathbb{Z} \cong \pi_1(V' \setminus \{p\}) \to \pi_1(U \setminus \{p\}) \to \pi_1(V \setminus \{p\}) \cong \mathbb{Z}$$

is an isomorphism, but the middle entry is trivial, a contradiction.

Second, we claim that every edge is identified with at most one other edge. If not, p instead has arbitrarily small neighborhoods U homeomorphic to the interior of the subspace of $\mathbb{R}^3$ given by the union of k triangles with a common edge. We then have



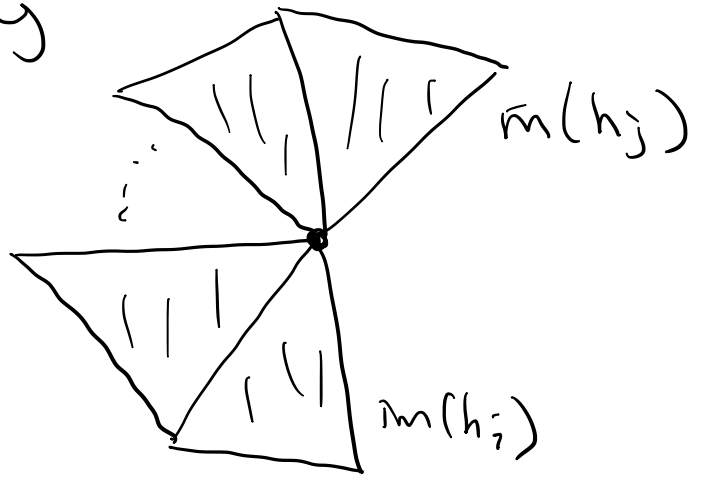
that the composite

$$\pi_1(U' \setminus \{p\}) \to \pi_1(V \setminus \{p\}) \to \pi_1(U \setminus \{p\})$$

is an isomorphism, the left and right entries are non-Abelian, and the middle is Abelian.

So far we have proven that h identifies edges in pairs, and it remains to show that the quotient space obtained from $\bigcup_{i=1}^{r} T_i$ by only doing so is homeomorphic to Σ. This amounts to showing that, if $\mathrm{im}(h_i) \cap \mathrm{im}(h_j)$ is a vertex of each, then there is a sequence of triangles sharing edges interpolating between

T_i and T_j. Similarly, assuming otherwise, one shows that the image p of the vertex has arbitrarily small open neighborhoods U such that $U - \{p\}$ is disconnected, leading to a contradiction as before.



Cor If Σ is a connected compact surface admitting a triangulation, then Σ is homeomorphic to the quotient of a polygonal region arising from a proper labeling scheme.

Proof By the theorem, Σ is obtained by identifying edges of $T_1, \dots, T_r$ in pairs. Then $r \geq 1$, and, without loss of generality, $m(h_1)$ and $m(h_2)$ intersect in the image of an edge, since otherwise Σ is not connected. Thus, Σ is obtained from $P_1, T_3, \dots, T_r$ by identifying edges in pairs, where P_1 is the four-sided polygonal region obtained by gluing T_1 and T_2 along the appropriate edge. If $r = 2$, we are finished. Otherwise, repeat the argument.

□