Last time

- Free variables and rank
- Rank and solution behavior
- Matrices as functions

We saw that an $m \times n$ matrix A determines a function $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ sending the vector $\vec{x}$ to the "matrix product" $A\vec{x}$, whose i^{th} entry is the dot product of $\vec{x}$ with the ith row of A. The system represented by A is captured by the equation

$$A\vec{x} = \vec{b}$$

We described the product $A\vec{x}$ above in terms of the rows of A. In terms of columns: the vector $A\vec{x}$ is obtained by scaling each column of A by the corresponding entry of $\vec{x}$ and adding the results, i.e.,

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \Rightarrow A\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n .$$

Ex $\begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 18 \end{bmatrix}$

$= \begin{bmatrix} 14 \\ 24 \end{bmatrix}$.

An expression $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ is called a <u>linear combination</u> of the vectors $\vec{v}_1, \ldots, \vec{v}_n$. So:

$A\vec{x} = \vec{b}$ has a solution	$\Longleftrightarrow$	$\vec{b}$ is a linear combination of the columns of A

The matrix product $A\vec{x}$ plays well with additions and scalar multiplications.

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} \quad \text{and} \quad A(c\vec{x}) = cA\vec{x}$$

OR

$$A(a\vec{x}+b\vec{y}) = aA\vec{x} + bA\vec{y}$$

Def A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** (or a linear transformation) if

$$T(a\vec{x}+b\vec{y}) = aT(\vec{x})+bT(\vec{y})$$

for every pair of vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$ and scalars a, b in $\mathbb{R}$.

Big idea

(1) The function determined by a matrix is a linear transformation.

(2) Every linear transformation is determined by a matrix.

Which matrix?

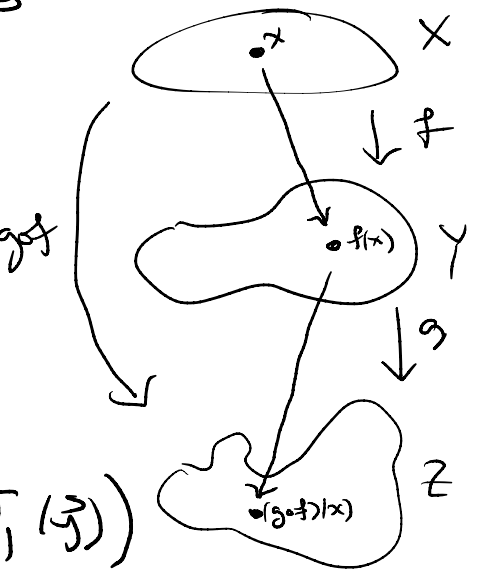
The jth column of the matrix corresponding to the linear transformation T is the vector $T\vec{e}_j$, where

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th}$$

i.e., it is the matrix

$$\begin{bmatrix} | & & | \\ T\vec{e}_1 & \cdots & T\vec{e}_n \\ | & & | \end{bmatrix}.$$

Given fuctions $f: X \to Y$ and $g: Y \to Z$, the composite is the fuctions $g \circ f: X \to Z$ defined by $(g \circ f)(x) = g(f(x))$. Applying this idea to linear transformations $T_1: \mathbb{R}^n \to \mathbb{R}^m$ and $T_2: \mathbb{R}^m \to \mathbb{R}^e$, we have that



$$(T_2 \circ T_1)(a\vec{x} + b\vec{y}) = T_2(T_1(a\vec{x} + b\vec{y}))$$

$$= T_2(aT_1(\vec{x}) + bT_1(\vec{y}))$$

$$= aT_2(T_1(\vec{x})) + bT_2(T_1(\vec{y}))$$

$$= a(T_2 \circ T_1)(\vec{x}) + b(T_2 \circ T_1)(\vec{y}),$$

So $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^2$ is also linear... so it corresponds to a matrix!