Last time

- Inverse of a 2x2 matrix
- Image and kernel

The kernel and image have properties in common.

(1) $\vec{0}$ is in both

(2) If $\vec{v}_1$ and $\vec{v}_2$ are in one, then so is $\vec{v}_1 + \vec{v}_2$.

(3) If $\vec{v}$ is in one, then so is $c\vec{v}$ for any number c.

A collection of vectors with these properties is called a (linear) <u>subspace</u>.

<u>Ex</u> $\{\vec{0}\}$ is a subspace.

<u>Ex</u> A line through $\vec{0}$ is a subspace.

<u>Ex</u> A plane through $\vec{0}$ is a subspace.

<u>Ex</u> $\mathrm{im}(T)$ is a subspace of $\mathbb{R}^m$ and $\ker(T)$ is a subspace of $\mathbb{R}^n$.

How can we understand what these spaces really are?

<u>Observation</u> Let $A = [\vec{v}_1 \cdots \vec{v}_n]$. A vector is in the image of A if and only if it is of the form

$$A\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n,$$

i.e., if it is a <u>linear combination</u> of the columns of A.

<u>Def</u> The <u>span</u> of a collection of vectors is the collection of all linear combinations of its vectors.

Ex The span of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ is $\mathbb{R}^2$.

Ex The span of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$ is the x-axis in $\mathbb{R}^2$.

So the image of A is the span of its column vectors. But this description can be very redundant.

Def Consider the list of vectors $\vec{v}_1, \ldots, \vec{v}_r$ in $\mathbb{R}^n$.

(1) The vector $\vec{v}_i$ is <u>redundant</u> if it is

a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_{i-1}$.

(2) The vectors $\vec{v}_1, \dots, \vec{v}_r$ are linearly independent if none are redundant.

(3) A linear relation among $\vec{v}_1, \dots, \vec{v}_r$ is an equation of the form

$$c_1\vec{v}_1 + \dots + c_r\vec{v}_r = \vec{0}.$$

It is trivial if $c_1 = c_2 = \dots = c_r = 0$ and nontrivial otherwise.

Having no redundant vectors is the same as having no nontrivial linear relations.