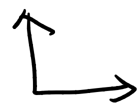
Last time

- More on linear independence and spanning
- Bases for subspaces

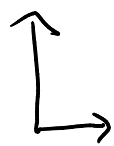
If $\{\vec{v}_1, \dots, \vec{v}_r\}$ is a basis for the subspace V, then any vector $\vec{v}$ in V can be written <u>uniquely</u> as a linear combination

$$\vec{v} = c_1\vec{v}_1 + \dots + c_r\vec{v}_r$$

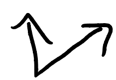
<u>Ex</u> $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



<u>Ex</u> $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{y}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$



<u>Ex</u> $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (y-x)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



As we saw last time,

columns of A linearly independent $\iff$ $A\vec{x}=\vec{0}$ has a unique solution $\iff \ker A = \{\vec{0}\} \iff rk(A)=n$

columns of A span $\mathbb{R}^m$ $\iff$ $A\vec{x}=\vec{b}$ has a solution for every $\vec{b}$ $\iff \operatorname{im} A = \mathbb{R}^m \iff rk(A)=m$

Combining these,

A is invertible

columns of A are a basis for $\mathbb{R}^m$ $\iff$ $A\vec{x}=\vec{b}$ has a unique solution for every $\vec{b}$ $\iff$ $\ker A = \{\vec{0}\}$ & $\operatorname{im} A = \mathbb{R}^m$ $\iff$ $rk(A) = m = n$

Let's try to find a basis for the kernel and image of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix}$$

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution to $A\vec{x}=\vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2-3x_4+4x_5 \\ x_2 \\ 4x_4-5x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= r\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + s\underbrace{\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} + t\underbrace{\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_3},$$

So $\ker(A) = \operatorname{Span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$. But there can be no relations among these vectors, since each has a nonzero entry where the others don't.

$$\Longrightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ is a basis for } \ker(A)$$

In particular, $\dim \ker(A) = 3$.

$$\boxed{\text{The dimension of } \ker(A) \text{ is the number of free variables. Basis vectors are obtained by setting one free variable equal to 1 and all others to 0}}$$

What about the image? We need to eliminate the redundant vectors from the columns of A. These correspond to the redundant columns in RREF(A), even though these columns span <u>different</u> subspaces.

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In our example, the second, third and fourth column are clearly redundant, since e.g.,

$$\begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently, the vector $\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$ is in the kernel of RRE(A). But this is the same as ker(A)! So the same relation holds for the columns of A.

So a basis for the image is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix} \right\}$$ and $\dim \operatorname{im}(A) = 2$.

The dimension of im(A) is the rank of A. A basis is given by the columns of A that become pivot columns in RREF(A)

Rank-nullity theorem If A is $m \times n$,

$$\boxed{\dim \operatorname{im}(A) + \dim \ker(A) = n}$$