Last time

- When is $[T]_{\mathcal{B}}$ diagonal?
- Orthonormality

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

<u>Ex</u> $\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

orthonormal vectors are
linearly independent

So a collection of linearly independent vectors is automatically a basis for its span, and a collection of m orthonormal vectors in $\mathbb{R}^m$ is automatically a basis for $\mathbb{R}^m$, called an <u>orthonormal basis</u>.

If $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_r\}$ is an orthonormal basis for V, then

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_r \cdot \vec{x} \end{bmatrix}.$$

Using orthonormal bases, we can generalize projection onto a line or plane to orthogonal projection onto any subspace V.

Specifically, given $\vec{x}$ in $\mathbb{R}^m$ and a subspace V, there are unique vectors $\vec{x}^{\perp}$ and $\vec{x}^{\parallel}$ with the following properties

(1) $\vec{x}^{\parallel}$ is in V

(2) $\vec{x}^{\perp}$ is orthogonal to every vector in V

(3) $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$.

The assignment $\vec{x} \longmapsto \vec{x}^{\parallel}$ is a linear transformation called proj_V.

Why can we do this? Choosing an

orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_r\}$ for V, we define

$$\vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_r \cdot \vec{x})\vec{u}_r$$

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}.$$

Conditions (1) and (2) are obviously satisfied, and condition (3) is also since given $\vec{v}$ in V, since

$$\vec{u}_i \cdot \vec{x}^{\perp} = \vec{u}_i \cdot \vec{x} - \vec{u}_i \cdot \vec{x}^{\parallel}$$

$$= \vec{u}_i \cdot \vec{x} - (\vec{u}_1 \cdot \vec{x})(\vec{u}_i \cdot \vec{u}_1) + \cdots + (\vec{u}_r \cdot \vec{x})(\vec{u}_i \cdot \vec{u}_r)$$

$$= \vec{u}_i \cdot \vec{x} - (\vec{u}_i \cdot \vec{x})(\vec{u}_i \cdot \vec{u}_i)$$

$$= \vec{u}_i \cdot \vec{x} - \vec{u}_i \cdot \vec{x}$$
$$= 0,$$

and a general vector in V is a linear combination of the $\vec{u}_i$.

$$\boxed{\mathrm{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_r \cdot \vec{x})\vec{u}_r}$$

This process relied on the existence of an orthonormal basis, which we justify next.