Last time

- Orthogonal $\Longrightarrow$ linearly independent
- $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_r\}$ ONB $\Longrightarrow$ $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_r \cdot \vec{x} \end{bmatrix}$
- Orthogonal projection onto V

$$\boxed{\operatorname{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_r \cdot \vec{x})\vec{u}_r}$$

For this to work, we need to be able to find an ONB for V.

Gram-Schmidt process

(0) Start with a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_r\}$ for the subspace V.

(1) Set $\vec{u}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|}$.

(2a) Write $\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$ with respect to Span$(\vec{u}_1)$.

(2b) Set $\vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{|\vec{v}_2^{\perp}|}$.

$\vdots$

(ja) Write $\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}$ with respect to $\mathrm{Span}(\vec{u}_1, \dots, \vec{u}_{j-1})$.

(jb) Set $\vec{u}_j = \dfrac{\vec{v}_j^{\perp}}{|\vec{v}_j^{\perp}|}$.

$\vdots$

Then the resulting vectors $\{\vec{u}_1, \dots, \vec{u}_r\}$ are orthonormal by construction, hence an orthonormal basis for V, since $\dim V = r$.

<u>Ex</u> Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. Then

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

(1) $\vec{u}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|} = \dfrac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(2a) $$\begin{aligned} \vec{v}_2^{\perp} &= \vec{v}_2 - \vec{v}_2^{\parallel} \\ &= \vec{v}_2 - \mathrm{proj}_{\mathrm{span}(\vec{u}_1)}(\vec{v}_2) \\ &= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 \\ &= \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

(2b) $\vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{|\vec{v}_2^{\perp}|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(3a) $\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel}$

$$= \vec{v}_3 - \mathrm{proj}_{\mathrm{span}(\vec{u}_1, \vec{u}_2)}(\vec{v}_3)$$

$$= \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

(3b) $\vec{u}_3 = \dfrac{\vec{v}_3^{\perp}}{|\vec{v}_3^{\perp}|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

The GS process begins with a basis $\{\vec{v}_1, \dots, \vec{v}_r\}$ and returns an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r\}$ defined recursively by

$$\boxed{\begin{aligned} \vec{u}_j &= \vec{v}_j^{\perp} / |\vec{v}_j^{\perp}| \\ \vec{v}_j^{\perp} &= \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j)\vec{u}_1 - \cdots - (\vec{u}_{j-1} \cdot \vec{v}_j)\vec{u}_{j-1} \end{aligned}}$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\} \to \boxed{\text{GS}} \to \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$