Last time

- Gram-Schmidt process

$$\underline{\text{Ex}} \quad \left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\} \to \boxed{\text{GS}} \to \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

In our example, we can summarize the relationship between our old and new bases like this:

$$\begin{bmatrix} 0 & 3 & 3 \\ 0 & 0 & 4 \\ 2 & 2 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}}_{R}$$

The matrices Q and R are special: the columns of Q are orthonormal, and R is upper triangular.

<u>QR factorization</u> An $m \times n$ matrix M with linearly independent columns may be factored uniquely as

$$M = QR$$

where Q is an $m \times n$ matrix with orthonormal columns and R is an $n \times n$ upper triangular matrix with positive diagonals.

To calculate the QR factorization:

(1) the columns of Q are obtained by applying Gram-Schmidt to the columns of M

(2) $R_{11} = |\vec{v}_1|$, $R_{jj} = |v_j^{\perp}|$ for $j > 2$, and $R_{ij} = \vec{u}_i \cdot \vec{v}_j$ for $i < j$.

We now consider what makes matrices like Q special.

<u>Def</u> A square matrix is called an <u>orthogonal</u> matrix if its columns are orthonormal.

<u>Ex</u> $\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

<u>Ex</u> $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

<u>Ex</u> $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

<u>Ex</u> $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

__Def__ The __transpose__ of A is the matrix A^T whose rows are the columns of A

A is orthogonal $\iff$ $A^{-1} = A^T$

Indeed, the (i,j) entry of A^TA is the dot product of the i^{th} row of A^T, which is the i^{th} column of A, with the j^{th} column of A. So if $A = [\vec{u}_1 \cdots \vec{u}_n]$, the (i,j) entry of A^TA is $\vec{u}_i \cdot \vec{u}_j$, and the resulting matrix is the identity precisely when

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Properties of transposes

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(AB)^T = B^T A^T$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

Orthogonal matrices preserve dot products:

$$\begin{aligned} A\vec{v}\cdot A\vec{w} &= (A\vec{v})^T A\vec{w} \\ &= \vec{v}^T A^T A\vec{w} \\ &= \vec{v}^T A^{-1} A\vec{w} \\ &= \vec{v}^T \vec{w} \\ &= \vec{v}\cdot\vec{w}. \end{aligned}$$

Dot product $\Longleftrightarrow$ length and angle

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

In particular, they preserve lengths of vectors, and this is actually equivalent, since

$$\vec{v}\cdot\vec{w} = \frac{1}{2}\left(|\vec{v}|^2 + |\vec{w}|^2 - |\vec{v}-\vec{w}|^2\right).$$

Characterizations of orthogonality

(1) The columns of A are an ONB
(2) $A^{-1} = A^T$
(3) $A^T A = I_n$
(4) A preserves length
(5) A preserves dot products (length & angle)