

Last time

- Proj_V minimizes distance
- Least squares solutions
- Normal equation

$$A^T A \vec{x} = A^T \vec{b}$$

Ex Let's find the line that best fits the data points (1,1), (2,3), (3,4)

Writing $y = ax + b$, we have the system

$$\left. \begin{array}{l} 1 = a + b \\ 3 = 2a + b \\ 4 = 3a + b \end{array} \right\} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 0 & 3 \end{array} \right] \begin{array}{l} a = 2 \\ a = 3/2 \end{array}$$

So the system is inconsistent. To find the LSS, we use the normal equations:

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

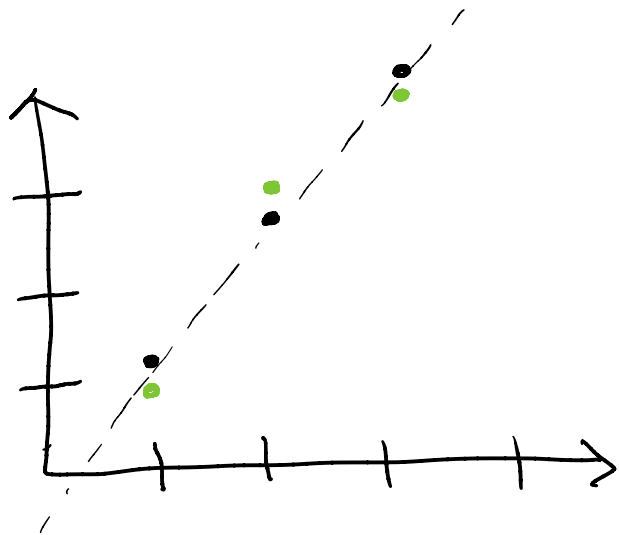
$$\left[\begin{array}{cc|c} 14 & 6 & 19 \\ 6 & 3 & 8 \end{array} \right]$$

$$= \begin{bmatrix} 19 \\ 8 \end{bmatrix}$$

↓

$$\left[\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 3 & -1 \end{array} \right] \quad \begin{array}{l} a = 3/2 \\ b = -1/3 \end{array}$$

The line of best fit is $y = \frac{3}{2}x - \frac{1}{3}$



Data	Line
(1, 1)	(1, 1 $\frac{1}{6}$)
(2, 3)	(2, 2 $\frac{2}{3}$)
(3, 4)	(3, 4 $\frac{1}{6}$)

In this example, there was a unique least squares solution because $\ker(A^T A) = \{\vec{0}\}$. Since $A^T A$ is square,

this happens precisely when $A^T A$ is invertible.

Unique least squares $A^T A$ is invertible if and only if $\ker(A) = \{\vec{0}\}$, in which case $A\vec{x} = \vec{b}$ has the unique LSS

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

Ex In our previous example,

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix}$$

$$(A^T A)^{-1} A^T \vec{b} = \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 19 \\ 8 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ -1/3 \end{bmatrix}.$$

To justify the claim about $A^T A$, it suffices to show that $\ker(A) = \ker(A^T A)$, but

$$\begin{aligned} A^T A \vec{x} = \vec{0} &\iff A \vec{x} \text{ is in } \ker(A^T) = m(A)^\perp \\ &\iff A \vec{x} \text{ is in } m(A) \text{ and } m(A)^\perp \\ &\iff A \vec{x} = \vec{0}. \end{aligned}$$

Goal Decide whether a matrix is invertible without having to row reduce.

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen that the determinant $ad - bc$ tells the whole story.

Recall A is invertible if and only if its columns are linearly independent.

For two vectors, this just means $\begin{bmatrix} a \\ c \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} b \\ d \end{bmatrix}$, and this is what the determinant detects.

On the other hand, three vectors are linearly dependent precisely when all three lie in a plane, which is to say that there is some vector orthogonal to all three.

Def The determinant of the 3×3 matrix $A = [\vec{u} \ \vec{v} \ \vec{w}]$ is the number $\det(A) = \vec{u} \cdot (\vec{v} \times \vec{w})$.

This number does the job since

$\det(A) = 0 \iff \vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$

\iff either \vec{v} and \vec{w} are parallel or \vec{u} lies in the plane spanned by \vec{v} and \vec{w}

$\iff \{\vec{u}, \vec{v}, \vec{w}\}$ is not linearly independent

$\iff A$ is not invertible.

If we unpack this formula, we get the following mnemonic, called Sarrus' rule:

$$\begin{array}{ccc|cc}
 a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
 a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
 a_{31} & a_{32} & a_{33} & a_{31} & a_{32}
 \end{array}$$

+
+
+

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
 - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}}_A \begin{array}{c} 1 \\ 4 \\ 7 \end{array} \begin{array}{c} 2 \\ 5 \\ 8 \end{array}$$

$$\det(A) = 50 + 84 + 96 - 105 - 48 - 80 \\
 = 230 - 233 = -3 \quad \text{Inventorke!}$$