Last time

- Goal: $\det(A) \neq 0 \iff A$ invertible
- 3×3 determinant

Sarrus:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$-\quad -\quad -\qquad\qquad +\quad +\quad +$

Ex

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}}_{A} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}$$

$$\det(A) = 50 + 84 + 96 - 105 - 48 - 80$$
$$= 230 - 233 = -3$$

Invertible!

We might hope to add up products of entries (with signs) in a general $n \times n$ matrix to define $\det(A)$ in general. How to decide which products and signs?

In the 2×2 case, we have the terms

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ \qquad\qquad\qquad\qquad -$$

For 3x3, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \qquad\qquad\qquad\qquad +$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \qquad\qquad\qquad\qquad -$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$- \qquad\qquad\qquad\qquad -$$

__Def__ A __pattern__ in an $n \times n$ matrix A is a choice of n entries of A such that there is exactly one entry in each row and column. Two entries in a pattern P are __inverted__ if one is above and to the right of the other. We say P is __odd__ or __even__ according to whether it has an odd or even number of inversions.

The <u>determinant</u> of A is

$$\det(A) = \sum \operatorname{sgn} P \cdot \operatorname{prod} P,$$

where the summation is over all patterns P, $\operatorname{prod} P$ is the product of the entries of P, and

$$\operatorname{sgn} P = \begin{cases} +1 & \text{if } P \text{ is even} \\ -1 & \text{if } P \text{ is odd}. \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 0$$

$$+$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 2$$

$$+$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 2$$

$$+$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 3$$

$$-$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 1$$

$$-$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 1$$

$$-$$

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad 7$$

$$\det(A) = -3 \cdot 2 \cdot 1 \cdot 8 \cdot 5 \cdot 2 = -480$$

Goal $\det(A) = 0 \iff A$ is not invertible

Idea We test invertibility with row reduction, so let's try to relate determinants and row operations.

For 2×2 matrices, this is easily done.

Matrix	Determinant
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$ad - bc$
$\begin{bmatrix} c & d \\ a & b \end{bmatrix}$	$cb - da = -(ad - bc)$
$\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$	$kad - kbc = k(ad - bc)$
$\begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$	$ad + kab - bc - kab = ad - bc$

<u>Guess</u> Let A be an $n \times n$ matrix.

(1) Swapping two rows of A changes the determinant by -1.

(2) Scaling a row by k scales the determinant by k.

(3) Adding a multiple of a row to another doesn't change the determinant

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Not only will this accomplish our goal, it gives us a method for calculating determinants.

Ex Let's calculate the determinant

of
$$A = \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[1+2]{\text{swap}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[1]{\text{subtract}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[3+4]{\text{swap}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} = B$$

So
$$\det(A) = (-1)^2 \det(B)$$
$$= (-1)^2 \cdot 1 \cdot 7 \cdot (-1)(-2)$$
$$= 14.$$