Last time

- Patterns and signs
- Definition of $\det(A) = \sum_P \text{sgn}(P)\,\text{prod}(P)$
- Upper triangular case

Goal $\det(A) = 0 \iff A$ is not invertible

Idea We test invertibility with row reduction, so let's try to relate determinants and row operations.

Based on the 2×2 case, we formulated the following:

<u>Guess</u> Let A be an $n \times n$ matrix.

(1) Swapping two rows of A changes the determinant by -1.

(2) Scaling a row by k scales the determinant by k.

(3) Adding a multiple of a row to another doesn't change the determinant.

If we can check this guess, then it will follow that

$$\det(A) = 0 \iff \det(RRE(F)) = 0,$$

since none of the row operations can change whether $\det(A)$ is zero.

But $RREF(A)$ is upper triangular with every diagonal entry either 0 or 1. So

$$\det(RREF(A)) = \begin{cases} 1 & RREF(A) = I_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\boxed{\det(A) \neq 0 \iff A \text{ is invertible}}$$

Our guess will follow from two important properties of determinants.

(A) The determinant is "multilinear", i.e., linear in each row separately. Specifically, given fixed vectors $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n$ in $\mathbb{R}^n$, the function

$$T(\vec{x}) = \det \begin{bmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{v}_{i-1} \rightarrow \\ \leftarrow \vec{x} \rightarrow \\ \leftarrow \vec{v}_{i+1} \rightarrow \\ \vdots \\ \leftarrow \vec{v}_n \rightarrow \end{bmatrix}$$

is linear. Indeed, for every pattern P, $\operatorname{prod} P$ is a product of $(n-1)$ fixed numbers (drawn from the $\vec{v}$'s) with a single entry of $\vec{x}$. This is obviously linear in $\vec{x}$, so $\operatorname{prod} P$ is, so $\det(A)$ is.

(B) The determinant is "alternating", i.e., $\det(A) = 0$ if two adjacent rows are equal. Indeed, for every pattern P, there is another pattern P' that interchanges the entries drawn from the duplicate rows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$P \qquad\qquad\qquad\qquad P'$$

The entries of P' drawn from these rows are inverted precisely if they were not inverted in P, and all other inversions are the same (because the rows are adjacent). Hence $\mathrm{prod}(P) = \mathrm{prod}(P')$ and $\mathrm{sgn}(P) = -\mathrm{sgn}(P')$, so the terms cancel.

Back to row operations.

(1) Swapping two rows

$$0 = \det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x}+\vec{y} \\ \vec{x}+\vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix} = \det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vec{x}+\vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix} + \det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{y} \\ \vec{x}+\vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix}$$

$$= \cancelto{0}{\det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix}} + \det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix} + \det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{y} \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix} + \cancelto{0}{\det\begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{y} \\ \vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix}}$$

If the rows aren't adjacent, swap adjacent rows until they are.

(2) Scaling a row

$$\det\begin{bmatrix}\vec{v}_1\\ \vdots\\ k\vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix} = k\cdot\det\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix}$$

(3) Adding a multiple of a row

$$\det\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}+k\vec{v}_j\\ \vdots\\ \vec{v}_n\end{bmatrix} = \det\begin{bmatrix}\vec{v}_1\\ \vdots\\ \vec{x}\\ \vdots\\ \vec{v}_n\end{bmatrix} + k\det\cancelto{0}{\begin{bmatrix}\vec{v}_j\\ \vdots\\ \vec{v}_j\\ \vdots\\ \vec{v}_n\end{bmatrix}}$$

As a final note, the determinant is also multilinear and alternating in the columns of the matrix.

Calculating det(A)

(1) Use row operations to turn A into a matrix B such that $\det(B)$ is easy (e.g., triangular).

(2) If you used s many row swaps and divided rows by scalars $k_1, \ldots, k_r$, then

$$\det(A) = (-1)^s k_1 \cdots k_r \det(B)$$

Using this technique, we can calculate the determinant of a product. It's easy to see that, if A' is obtained from A using a row operation, then $A'B$ is obtained from AB using the same row operation. If A is invertible, then, we have

$$[A \mid I_n] \longrightarrow [I_n \mid A^{-1}]$$

using s swaps and dividing by $k_1, \dots, k_r$. Therefore,

$$[AB \mid B] \longrightarrow [B \mid A^{-1}B]$$

using the same operations. Then

$$\det(AB) = (-1)^s k_1 \cdots k_r \det(B)$$

$$= (-1)^s k_1 \cdots k_r \det(I_n)\det(B)$$

$$= \det(A)\det(B).$$

If A is not invertible, then neither is AB; indeed, since A is $n \times n$

$$A \text{ not invertible} \iff \operatorname{im}(A) \neq \mathbb{R}^n,$$

but $\operatorname{im}(AB)$ is contained in $\operatorname{im}(A)$, so $\operatorname{im}(AB) \neq \mathbb{R}^n$. In this case,

$$\det(AB) = 0 = 0 \cdot \det(B) = \det(A)\det(B)$$

$$\boxed{\det(AB) = \det(A)\det(B)}$$