

## Last time

- Patterns and signs
- Definition of  $\det(A) = \sum_P \text{sgn}(P) \text{Prod}(P)$
- Upper triangular case

Goal  $\det(A) = 0 \iff A$  is not invertible

Idea We test invertibility with row reduction, so let's try to relate determinants and row operations.

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Based on the  $2 \times 2$  case, we formulated the following:

Guess Let  $A$  be an  $n \times n$  matrix.

(1) Swapping two rows of  $A$  changes the determinant by  $-1$ .

(2) Scaling a row by  $k$  scales the determinant by  $k$ .

(3) Adding a multiple of a row to another doesn't change the determinant.

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If we can check this guess, then it will follow that

$\det(A) = 0 \iff \det(\text{RREF}(A)) = 0$ ,  
since none of the row operations can  
change whether  $\det(A)$  is zero.

But  $\text{RREF}(A)$  is upper triangular  
with every diagonal entry either 0  
or 1. So

$$\det(\text{RREF}(A)) = \begin{cases} 1 & \text{RREF}(A) = I_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\det(A) \neq 0 \iff A \text{ is invertible}$$

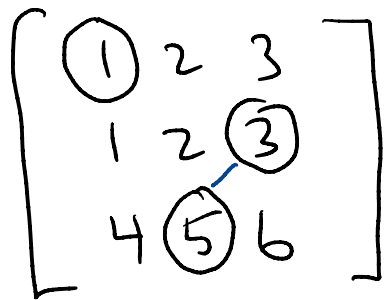
Our guess will follow from two important properties of determinants.

(A) The determinant is "multilinear," i.e., linear in each row separately. Specifically, given fixed vectors  $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n \in \mathbb{R}^n$ , the function

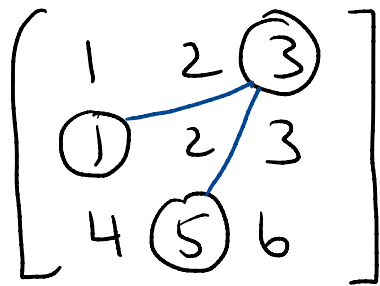




(B) The determinant is "alternating", i.e.,  $\det(A) = 0$  if two adjacent rows are equal. Indeed, for every pattern  $P$ , there is another pattern  $P'$  that interchanges the entries drawn from the duplicate rows:



$P$



$P'$

The entries of  $P'$  drawn from these rows are inverted precisely if they were not inverted in  $P$ , and all other inversions are the same (because the rows are adjacent). Hence  $\text{prod}(P) = \text{prod}(P')$  and  $\text{sgn}(P) = -\text{sgn}(P')$ , so the terms cancel.

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Back to row operations.

(1) Swapping two rows

$$0 = \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ x^u + y^u \\ \vdots \\ x^v + y^v \\ \vdots \\ \underline{z}_n \end{bmatrix} = \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ x^u \\ \vdots \\ y^v \\ \vdots \\ \underline{z}_n \end{bmatrix} + \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ y^u \\ \vdots \\ x^v \\ \vdots \\ \underline{z}_n \end{bmatrix}$$

$$= \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ x^u \\ \vdots \\ x^v \\ \vdots \\ \underline{z}_n \end{bmatrix} + \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ x^u \\ \vdots \\ y^v \\ \vdots \\ \underline{z}_n \end{bmatrix} + \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ y^u \\ \vdots \\ x^v \\ \vdots \\ \underline{z}_n \end{bmatrix} + \det \begin{bmatrix} \underline{z}_1 \\ \vdots \\ y^u \\ \vdots \\ y^v \\ \vdots \\ \underline{z}_n \end{bmatrix}$$

If the rows aren't adjacent, swap adjacent rows until they are.

(2) Scaling a row

$$\det \begin{bmatrix} \vec{r}_1 \\ \vdots \\ k\vec{r}_i \\ \vdots \\ \vec{r}_n \end{bmatrix} = k \cdot \det \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{bmatrix}$$

(3) Adding a multiple of a row

$$\det \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i + k\vec{r}_j \\ \vdots \\ \vec{r}_n \end{bmatrix} = \det \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{bmatrix} + k \det \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_n \end{bmatrix} \rightarrow 0$$

As a final note, the determinant is also multilinear and alternating in the columns of the matrix.

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### Calculating $\det(A)$

(1) Use row operations to turn  $A$  into a matrix  $B$  such that  $\det(B)$  is easy (e.g., triangular).

(2) If you used  $s$  many row swaps and divided rows by scalars  $k_1, \dots, k_r$ , then

$$\det(A) = (-1)^s k_1 \dots k_r \det(B)$$

Using this technique, we can calculate the determinant of a product. It's easy to see that, if  $A'$  is obtained from  $A$  using a row operation, then  $A'B$  is obtained from  $AB$  using the same row operation. If  $A$  is invertible, then, we have

$$[A | I_n] \rightarrow [I_n | A^{-1}]$$

using  $s$  swaps and dividing by  $k_1, \dots, k_r$ .  
Therefore,

$$[AB|B] \rightarrow [B|A^{-1}B]$$

using the same operations. Then

$$\begin{aligned}\det(AB) &= (-1)^{s_{k_1 \dots k_r}} \det(B) \\ &= (-1)^{s_{k_1 \dots k_r}} \det(I_n) \det(B) \\ &= \det(A) \det(B).\end{aligned}$$

If  $A$  is not invertible, then neither is  $AB$ ; indeed, since  $A$  is  $n \times n$

$$A \text{ not invertible} \iff \text{im}(A) \neq \mathbb{R}^n,$$



but  $\text{im}(AB)$  is contained in  $\text{im}(A)$ ,  
so  $\text{im}(AB) \neq \mathbb{R}^n$ . In this case,

$$\det(AB) = 0 = 0 \cdot \det(B) = \det(A)\det(B)$$

$$\det(AB) = \det(A)\det(B)$$