Last time

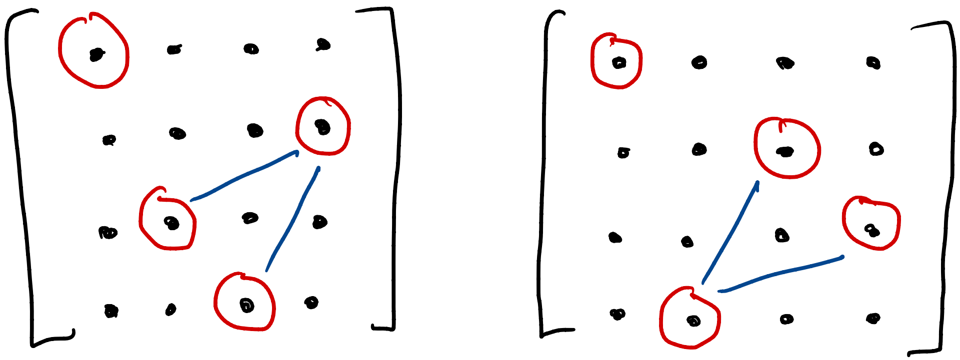
- $\det(A) \neq 0 \iff A$ invertible
- Row operations and determinants
- $\det(A)$ multilinear and alternating
- Calculating determinants
- $\det(AB) = \det(A)\det(B)$

From this, it follows that

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

We also have the relationship to transposes:

$$\boxed{\det(A) = \det(A^T)}$$



P $\qquad$ P^T

$\mathrm{prod}(P) = \mathrm{prod}(P^T)$ $\qquad$ $\mathrm{sign}(P) = \mathrm{sign}(P^T)$

We return to Sarrus' rule:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$- \quad - \quad - \qquad\qquad + \quad + \quad +$$

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \\ &\quad + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \end{aligned}$$

Laplace expansion Let A_{ij} be the $(n-1)\times(n-1)$ matrix obtained by omitting the i^{th} row and j^{th} column from A.

(1) Expansion along row i:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

(2) Expansion down column j:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Ex

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix} \qquad \left(\begin{bmatrix} + & - & + & \\ - & + & - & \cdots \\ + & - & + & \\ & \vdots & & \ddots \end{bmatrix} \right)$$

Expanding down the 2^{nd} column:

$$\det(A) = \cancelto{0}{-0 \cdot \det \begin{bmatrix} 9 & 3 & 0 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}} + 1 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$$

$$- 2 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 5 & 0 & 3 \end{bmatrix} + \cancelto{0}{0 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 9 & 2 & 0 \end{bmatrix}}$$

$$= 5 \det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 9 & 2 \end{bmatrix}$$

$$- 2 \left(2 \det \begin{bmatrix} 9 & 3 \\ 5 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 9 & 3 \end{bmatrix} \right)$$

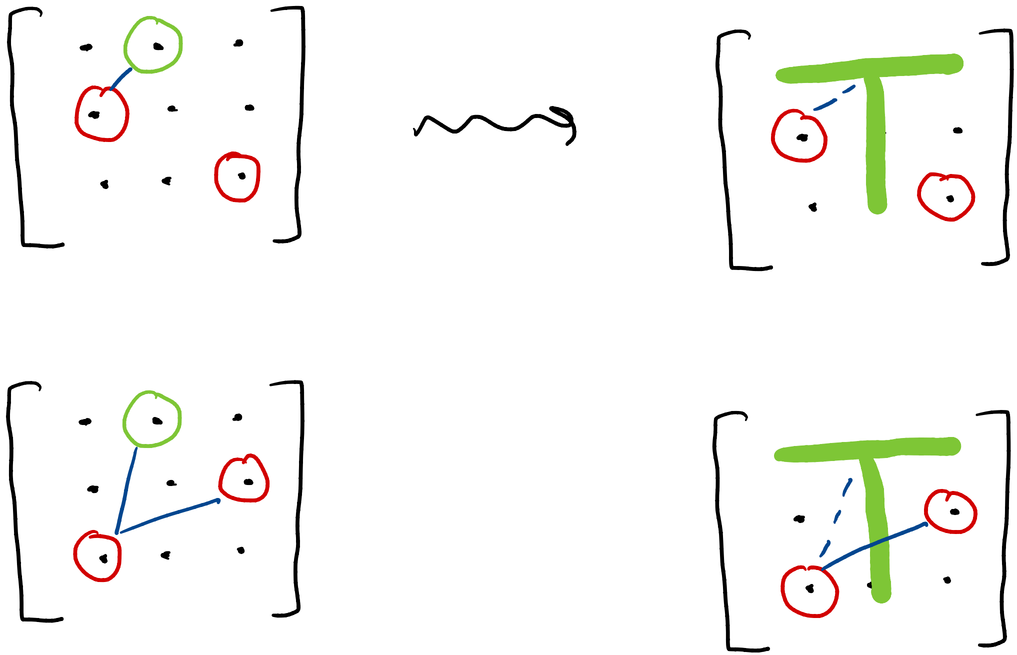
$$= 5(-4) + 3(-7) - 4(-15) - 6(-6)$$

$$= -20 - 21 + 60 + 36$$

$$= 55$$

Why does this work? For every entry in the i^{th} row, collect all the patterns involving that entry, and factor out that entry to get the determinant of the smaller matrix, up to a sign.

accounting for the lost inversions.



Using the determinant, a general system with invertible coefficient matrix can be solved explicitly.

Cramer's rule If A is an invertible $n\times n$ matrix, then the solution to $A\vec{x}=\vec{b}$ is

$$\vec{x} = \frac{1}{\det(A)}\begin{bmatrix} \det(A_{\vec{b},1}) \\ \vdots \\ \det(A_{\vec{b},n}) \end{bmatrix},$$

where $A_{\vec{b},i}$ is obtained from A by replacing the i^{th} column with $\vec{b}$.

Why? If $A = [\vec{v}_1 \cdots \vec{v}_n]$, then

$$\det(A_{\vec{b},i}) = \det\left(\left[\vec{v}_1 \cdots \vec{b} \cdots \vec{v}_n\right]\right)$$

$$= \det\left(\left[\vec{v}_1 \cdots A\vec{x} \cdots \vec{v}_n\right]\right)$$

$$= \det\left(\left[\vec{v}_1 \cdots (x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) \cdots \vec{v}_n\right]\right)$$

$$= x_1 \det\left(\left[\vec{v}_1 \cdots \vec{v}_1 \cdots \vec{v}_n\right]\right) + \cdots$$

$$+ x_n \det\left(\left[\vec{v}_1 \cdots \vec{v}_n \cdots \vec{v}_n\right]\right)$$

$$= x_i \det\left(\left[\vec{v}_1 \cdots \vec{v}_i \cdots \vec{v}_n\right]\right)$$

$$= x_i \det(A).$$

Cramer's rule lets us write down a formula for the inverse in general. If $A^{-1} = [\vec{w}_1 \cdots \vec{w}_n]$, then $A\vec{w}_j = \vec{e}_j$, since $AA^{-1} = I_n$. So

$$\vec{w}_j = \frac{1}{\det(A)} \begin{bmatrix} \det(A_{\vec{e}_{j,1}}) \\ \vdots \\ \det(A_{\vec{e}_{j,n}}) \end{bmatrix}$$

But, by Laplace expansion,

$$\det\left(A_{\vec{e}_{j,i}}\right) = \det\begin{bmatrix} | & & 0 & & | \\ \vec{v}_1 & \cdots & \vdots \; 1 \; \vdots & \cdots & \vec{v}_n \\ | & & 0 & & | \end{bmatrix}$$

$$= (-1)^{i+j} \det A_{j,i}$$

$$\boxed{\text{The } (i,j) \text{ entry of } A^{-1} \text{ is } \quad (-1)^{i+j}\,\frac{\det(A_{j,i})}{\det(A)}}$$