Last time

- Determinant of A^{-1} and A^T
- Laplace expansion
- Cramer's rule and inverses

Diagonal matrices are good because

(1) they're easy to work with

(2) their geometric meaning is transparent

(3) the systems of equations they represent are "uncoupled", so easy to solve

Matrices that are similar to diagonal matrices are just as good.

__Def__ An $n\times n$ matrix A is __diagonalizable__ __if A is similar to a diagonal matrix.__

In other words, $S^{-1}AS$ is diagonal for some invertible $n\times n$ matrix S, which is to say there is a basis $\mathcal{B}$ (the columns of S) for $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix of A is diagonal.

What does this mean, concretely?

Suppose

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D$$

In this case,

$$\begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} = AS = SD = \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}.$$

<u>Def</u> An <u>eigenvector</u> of A is a vector $\vec{v} \neq \vec{0}$ such that

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ, called the eigenvalue associated to $\vec{v}$. A basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ is called an eigenbasis for A if each $\vec{v}_i$ is an eigenvector of A.

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diagonalizable $\Longleftrightarrow$ an eigenbasis exists

More specifically, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent eigenvectors of A

with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

<u>Ex</u> Every nonzero vector is an eigenvector for I_n with eigenvalue 1.

<u>Ex</u> Any nonzero vector in V is an eigenvector for proj_V with eigenvalue 1.
Any nonzero vector in $V^\perp$ is an

eigenvector with eigenvalue 0.

Ex An eigenvector with eigenvalue 0 is the same thing as a nonzero vector in $\ker(A)$.

$$\boxed{A \text{ is invertible} \iff \begin{array}{c} 0 \text{ is not an} \\ \text{eigenvalue of } A \end{array}}$$

Ex Unless θ is a multiple of $180°$, rotation by θ has no eigenvectors.

<u>Ex</u> The eigenvalues of reflection across a line L in $\mathbb{R}^2$ are 1 (for a vector in L) and -1 (for a vector in $L^\perp$).

<u>Ex</u> If is A is orthogonal, then $|A\vec{v}| = |\vec{v}|$, so, if $A\vec{v} = \lambda\vec{v}$, then

$$|\vec{v}| = |A\vec{v}| = |\lambda\vec{v}| = |\lambda||\vec{v}|,$$

so the only possible eigenvalues of A are ± 1.