Last time

- Diagonalizability
- Eigenvalues & eigenvectors
- Examples

Q How to find eigenvalues/vectors?

Ex Let's find the eigenvalues/vectors of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

The vector $\vec{v}$ is an eigenvector with eigenvalue λ if and only if

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{v} = \lambda \vec{v} \iff \begin{array}{l} \vec{v}_1 + 2\vec{v}_2 = \lambda \vec{v}_1 \\ 4\vec{v}_1 + 3\vec{v}_2 = \lambda \vec{v}_2 \end{array}$$

$$\iff \begin{array}{l} (1-\lambda)\vec{v}_1 + 2\vec{v}_2 = 0 \\ 4\vec{v}_1 + (3-\lambda)\vec{v}_2 = 0 \end{array}$$

$\iff$ $\vec{v}$ is in the kernel of

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

There is no such $\vec{v}$ unless this matrix is non-invertible, which we can check with the determinant:

$$\det\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 8$$
$$= \lambda^2 - 3\lambda - \lambda + 3 - 8$$
$$= \lambda^2 - 4\lambda - 5$$
$$= (\lambda - 5)(\lambda + 1)$$

so the matrix is not invertible if and only if $\lambda = 5$ or $\lambda = -1$, i.e., these are the eigenvalues.

To find corresponding eigenvectors, we have to solve each system.

$\underline{\lambda = 5}$ $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{\text{Check}}$: $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\underline{\lambda = -1}$ $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\underline{\text{Check}}$: $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So $\{\vec{v}_1, \vec{v}_2\}$ is an eigenbasis, and A is diagonalizable. To check:

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S^{-1}AS = B \iff AS = SB$$

$$AS = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 10 & 1 \end{bmatrix}$$

$$SB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 10 & 1 \end{bmatrix}$$

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Let's formalize the ideas in this example.

Def The characteristic equation of the $n\times n$ matrix A is the equation

$$\det(A-\lambda I_n)=0,$$

viewed as an equation in the variable λ.

$$\det(A-\lambda I_n)=0 \iff A-\lambda I_n \text{ is not invertible}$$
$$\iff \ker(A-\lambda I_n)\neq\{\vec{0}\}$$
$$\iff (A-\lambda I_n)\vec{v}=\vec{0} \text{ has a nonzero solution}$$

$\iff A\vec{v} = \lambda\vec{v}$ has a nonzero solution

$\iff \lambda$ is an eigenvalue of A

The eigenvalues of A are exactly the solutions of the characteristic equation.

<u>Ex</u> If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, then

$$\det(A-\lambda I_n) = \det\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(6-\lambda),$$

So the eigenvalues of A are $\lambda = 1, 4, 6$.

$$\boxed{\text{The eigenvalues of a triangular matrix are the diagonal entries}}$$

Ex If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$$

Def The *trace* of an $n\times n$ matrix is the sum of its diagonal entries.

The solutions of

$$\lambda^2 - tr(A)\lambda + det(A) = 0$$

are $$\lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4det(A)}}{2}.$$

These are real numbers if and only $tr(A)^2 \geq 4det(A)$.

Ex The characteristic equation of

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is $\lambda^2+1=0$, which

has no (real) solutions, so this matrix has no (real) eigenvalues.

In the examples so far, $\det(A-\lambda I_n)$ is a polynomial of degree n. This is true in general, since Prod(P) is a polynomial of degree $\leq n$, with equality for the diagonal pattern.

Def The characteristic polynomial of A is $f_A(\lambda) = \det(A - \lambda I_n)$.

This polynomial has

- degree n
- leading coefficient $(-1)^n$
- second coefficient $(-1)^{n-1}\operatorname{tr}(A)$
- constant coefficient $\det(A)$.

The eigenvalues of A are the roots of $f_A(\lambda)$.