## Last time

- Calculating eigenvalues
- Characteristic equation/polynomial

$$f_A(\lambda) = \det(A - \lambda I_n) = 0$$

- Eigenvalues might not exist

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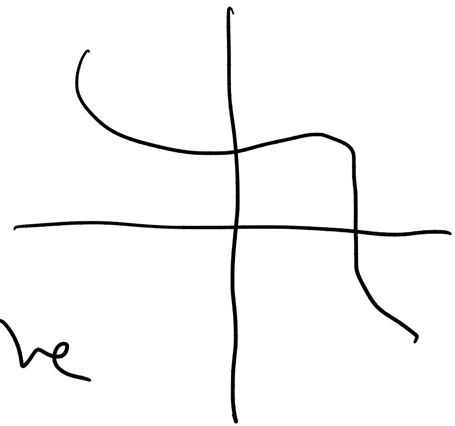
**Q** How many eigenvalues are there?

The fact that $f_A(\lambda)$ is a polynomial helps us approach this question.

(1) If $n$ is odd, then $A$ has at least one eigenvalue. Indeed,

$$\lim_{\lambda \to \pm\infty} f_A(\lambda) = \lim_{\lambda \to \pm\infty} (-1)^n \lambda^n$$

$$= \mp \infty$$



So the graph of $f_A(\lambda)$ crosses the $\lambda$ axis by the intermediate value theorem.

(2) An eigenvalue might occur "with multiplicity."

Ex

If $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$, then

$f_A(\lambda) = (1-\lambda)^3(2-\lambda)^2$, so $\lambda = 1$ "occurs three times" and $\lambda = 2$ "occurs twice".

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Def If $\lambda_0$ is an eigenvalue of $A$, the algebraic multiplicity of $\lambda_0$ is the natural number k such that

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

where $g(\lambda)$ is a polynomial with $g(\lambda_0) \neq 0$

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So an $n \times n$ matrix has <u>at most</u> $n$ eigenvalues, counted with multiplicity.

If $A$ has $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct), so that

$$f_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda),$$

then

$$tr(A) = \lambda_1 + \cdots + \lambda_n$$

$$\det(A) = \lambda_1 \cdots \lambda_n.$$

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We now have a good strategy for determining the eigenvalues of an $n \times n$ matrix $A$: solve the characteristic equation $\det(A - \lambda I_n) = 0$.

What about eigenvectors?

<u>Def</u> The subspace spanned by the eigenvectors of $A$ with eigenvalue $\lambda$ is called the <u>eigenspace</u> associated to $\lambda$, i.e.,

$$E_\lambda = \ker(A - \lambda I_n).$$

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Every nonzero vector in $E_\lambda$ is an eigenvector with eigenvalue $\lambda$ (and vice versa).

Ex As we saw, the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are $\lambda = 5, -1$, and

$$E_5 = \ker \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$E_{-1} = \ker \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

Ex Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Since $A$ is upper triangular, its eigenvalues are $\lambda = 0, 1$. To find the eigenspaces,

$$E_1 = \ker(A - I_3) = \ker\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \ker\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$E_0 = \ker(A - 0 \cdot I_3) = \ker\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \ker\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

Since $E_1$ and $E_0$ are both 1 dimensional, it is only possible to find 2 linearly independent eigenvectors for $A$, so there is no eigenbasis, i.e., $A$ is not diagonalizable.

<u>Def</u> The <u>geometric multiplicity</u> of $\lambda_0$ as an eigenvalue of $A$ is the dimension of the eigenspace associated to $\lambda_0$ ($\dim E_{\lambda_0}$).

In the last example, we had $f_A(\lambda) = \lambda(1-\lambda)^2$, so

| $\lambda$ | alg. mult. | geom. mult. |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 2 | 1 |