Last time

- Eigenspaces E_λ
- Geometric multiplicity $\dim(E_\lambda)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

λ	alg. mult.	geom. mult.
0	1	1
1	2	1

This matrix is <u>not</u> diagonalizable because the geometric multiplicity of $\lambda = 1$ is too small.

The following three conditions are equivalent for an $n \times n$ matrix A.

(1) A is diagonalizable

(2) A has n eigenvalues (not necessarily distinct) whose algebraic and geometric multiplicities coincide

(3) The dimensions of the eigenspaces of A add up to n.

The main ingredient here is:

$$\boxed{\text{Eigenvectors for distinct eigenvalues are linearly independent}}$$

Why? Suppose $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors for distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. If $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly dependent, then there is a <u>first</u> redundant vector $\vec{v}_i$.

Since $\vec{v}_i$ is redundant,

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}.$$

Applying the matrix $A - \lambda_i I_n$ gives

$$0 = (A-\lambda_i I_n)\vec{v}_i = c_1(A-\lambda_i I_n)\vec{v}_1 + \cdots$$

$$= c_1\left[A\vec{v}_i - \lambda_1\vec{v}_1\right] + \cdots$$

$$= c_1\left[\lambda_i\vec{v}_i - \lambda_1\vec{v}_1\right] + \cdots$$

$$= c_1(\lambda_i - \lambda_1)\vec{v}_1 + \cdots$$

$$\Rightarrow c_1(\lambda_i - \lambda_1)\vec{v}_1 + \cdots + c_{i-1}(\lambda_i - \lambda_{i-1})\vec{v}_{i-1} = 0$$

This is linear relation among the first $i-1$ vectors, so $\vec{v}_i$ wasn't the first redundant vector after all!

$$\boxed{\text{If } A \text{ has } n \text{ distinct eigenvalues, then } A \text{ is diagonalizable}}$$

Diagonalizing a matrix A

(1) Solve the characteristic equation to find the eigenvalues of A

(2) For each eigenvalue, find a basis for $E_\lambda = \ker(A - \lambda I_n)$.

(3) If the dimensions of the eigenspaces do not add up to n, then A is not diagonalizable.

(4) Otherwise, $S^{-1}AS = B$, where

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

the $\vec{v}_i$ are the basis vectors from step (2), and λ_i is the eigenvalue of $\vec{v}_i$.

Ex

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ -4 & -\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} -\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= -\lambda(1-\lambda)^2 \qquad \lambda = 0, 1$$

$$E_0 = \ker\begin{bmatrix}1&0&0\\-4&0&2\\0&0&1\end{bmatrix} = \ker\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

$$E_1 = \ker\begin{bmatrix}0&0&0\\-4&-1&2\\0&0&0\end{bmatrix} = \ker\begin{bmatrix}-4&-1&2\\0&0&0\\0&0&0\end{bmatrix}$$

$-4x_1 - x_2 + 2x_3 = 0$

x_2, x_3 free

$$= \mathrm{Span}\left(\begin{bmatrix}1\\-4\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right)$$

λ	alg. mult.	geom. mult.
0	1 $=$	1
1	2 $=$	2
		3 ✓

So A is diagonalizable: $S^{-1}AS = B$, where

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Q How badly can diagonalizability fail?

Ex $A = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ The only eigenvalue is $\lambda = 5$

$$E_5 = \ker \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

λ	alg. mult.	geo. mult.
5	4	1

Similarly, the $n \times n$ matrix

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

has eigenvalue λ with algebraic multiplicity n and geometric multiplicity 1.

Combining these, we can build custom non-diagonalizable matrices.

Ex A 6×6 matrix with

λ	alg. mult.	geo. mult.
2	3	2
3	3	1

$$\begin{bmatrix} \boxed{2} & & & & & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & 0 & \\ & & \boxed{\begin{matrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{matrix}} \\ 0 & & & & & \end{bmatrix}$$