Last time

- Diagonalization algorithm

A last word: similar matrices have the same characteristic polynomial, the same eigenvalues with the same algebraic and geometric multiplicities, and the same determinant and trace. Most of this comes down to the fact that

$A = SBS^{-1} \Rightarrow A - \lambda I_n = S(B - \lambda I_n)S^{-1}$.

We've spent lots of time investigating two types of special basis: orthonormal bases and eigenbases.

Q When can we have both?

Def We say A is orthogonally diagonalizable if there exists an orthonormal eigenbasis for A, i.e., if there is an orthogonal matrix S such that $S^{-1}AS$ is diagonal.

Ex Diagonal matrices

Ex Orthogonal projection

Ex Reflection

All three types are symmetric, and this is no coincidence. If $S^{-1}AS$ is diagonal, then

$$S^{-1}AS = (S^{-1}AS)^T$$
$$= S^TA^T(S^{-1})^T$$
$$= S^{-1}A^T(S^T)^T$$
$$= S^{-1}A^TS \implies A = A^T.$$

The converse is also true.

<u>Spectral theorem</u> A is orthogonally diagonalizable if and only if A is symmetric.

<u>Ex</u> $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$ $\qquad f_A(\lambda) = \lambda^2 - 11\lambda + 24$

$$\lambda = (\lambda - 3)(\lambda - 8)$$

$$E_3 = \ker \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$E_8 = \ker \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = \ker \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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Since E_3 is orthogonal to E_8, we get an ONEB by choosing a unit vector from each:

$$\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

This example illustrates a general fact.

If A is symmetric, then eigenvectors for distinct eigenvalues are orthogonal

Why? Suppose $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors for λ_1 and λ_2, respectively. We'd like to say that $\vec{v}_1 \cdot \vec{v}_2 = \vec{0}$. We have that

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1 \cdot A\vec{v}_2 = \vec{v}_1 \cdot \lambda_2 \vec{v}_2 = \lambda_2(\vec{v}_1 \cdot \vec{v}_2).$$

But, on the other hand,

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = (Av_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1) \cdot \vec{v}_2 = \lambda_1(\vec{v}_1 \cdot \vec{v}_2).$$

Subtracting these, we have

$$(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0.$$

Since $\lambda_1 \neq \lambda_2$, $\vec{v}_1 \cdot \vec{v}_2 = 0$.

So far, this shows that a symmetric matrix is orthogonally diagonalizable if it is diagonalizable. It remains to show that symmetric matrices are diagonalizable, which we omit.

Ex Let's orthogonally diagonalize

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$f_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \det\begin{bmatrix} 1 & 1 \\ 1 & 1-\lambda \end{bmatrix} + \det\begin{bmatrix} 1 & 1-\lambda \\ 1 & 1 \end{bmatrix}$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) + \lambda + \lambda$$

$$= -\lambda^3 + \lambda^2 + 2\lambda^2 - 2\lambda + 2\lambda$$

$$= -\lambda^2(\lambda - 3) \qquad \lambda = 0, 3$$

$$E_0 = \ker\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$E_3 = \ker\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{Span}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

<u>Sanity check</u>: $\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0.$

To get an ONEB, we apply Gram-Schmidt to E_0.

$$\vec{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\vec{v}_2^{\perp} = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

ONB for E_0: $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$

ONB for E_3: $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

ONEB: $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$

Orthogonal diagonalization

(0) If A is not symmetric, stop.

(1) Find a basis for each E_λ.

(2) Apply Gram-Schmidt obtain an ONB for each E_λ.

(3) If $\vec{u}_1, \ldots, \vec{u}_n$ are the vectors from (2), then the matrix

$$S = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$$

is orthogonal, and $S^{-1}AS$ is diagonal.