Last time

- Singular value decomposition

Ex $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ $A^TA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$f_{A^TA}(\lambda) = \lambda^2 - tr(A^TA)\lambda + det(A^TA)$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (\lambda - 1)(\lambda - 3)$$

$\lambda = 1, 3$

$\sigma_1 = \sqrt{3}$ $\sigma_2 = 1$

$$E_3 = \ker\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \ker\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_1 = \ker\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A\vec{v}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad A\vec{v}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The vector $\vec{u}_3$ is any vector that makes $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ an ONB, i.e., it is any unit vector orthogonal to $\vec{u}_1$ & $\vec{u}_2$.

$$\ker \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \mathrm{Span} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

We now consider another application.

Ex Suppose the populations of coyotes and rabbits are given as a function of time t (in years, say) by

$$c(t+1) = 0.86c(t) + 0.08r(t)$$

$$r(t+1) = -0.12c(t) + 1.14r(t)$$

which we can write as $\vec{x}_{t+1} = A\vec{x}_t$, where

$$\vec{x}_t = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} \qquad A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}.$$

So the population at time t is determined by the initial population by

$$\vec{x}_t = A^t \vec{x}_0$$

$$\begin{tikzcd} \vec{x}_0 \arrow[d, "A"] \\ \vec{x}_1 \arrow[d, "A"] \\ \vec{x}_2 \arrow[d, "A"] \\ \vdots \end{tikzcd}$$

We'd like an explicit formula for $\vec{x}_t$ as a function of t. For some initial conditions, this is easy.

(1) If $\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$, then

$$\vec{x}_1 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix} = 1.1 \begin{bmatrix} 100 \\ 300 \end{bmatrix}$$

$$\Longrightarrow \vec{x}_1 = 1.1\,\vec{x}_0$$

$$\Longrightarrow \vec{x}_t = (1.1)^t\,\vec{x}_0,$$

i.e., $c(t) = 100(1.1)^t$ and $r(t) = 300(1.1)^t$.

Both populations grow exponentially at a rate of 10%.

(2) If $\vec{x}_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, then

$$\vec{x}_1 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = 0.9 \begin{bmatrix} 200 \\ 100 \end{bmatrix},$$

so, as before, $\vec{x}_t = (0.9)^t\,\vec{x}_0$, i.e.,

$c(t) = 200(0.9)^t$ and $r(t) = 100(0.9)^t$.

Both populations decay exponentially.

(3) If $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$, things don't work out as nicely. But

$$\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = 2\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$$= 2\vec{v} + 4\vec{w}$$

$$\Longrightarrow \vec{x}_t = A^t \vec{x}_0$$

$$= A^t(2\vec{v} + 4\vec{w})$$

$$= 2A^t\vec{v} + 4A^t\vec{w}$$

$$= 2(1.1)^t \vec{v} + 4(0.9)^t \vec{w},$$

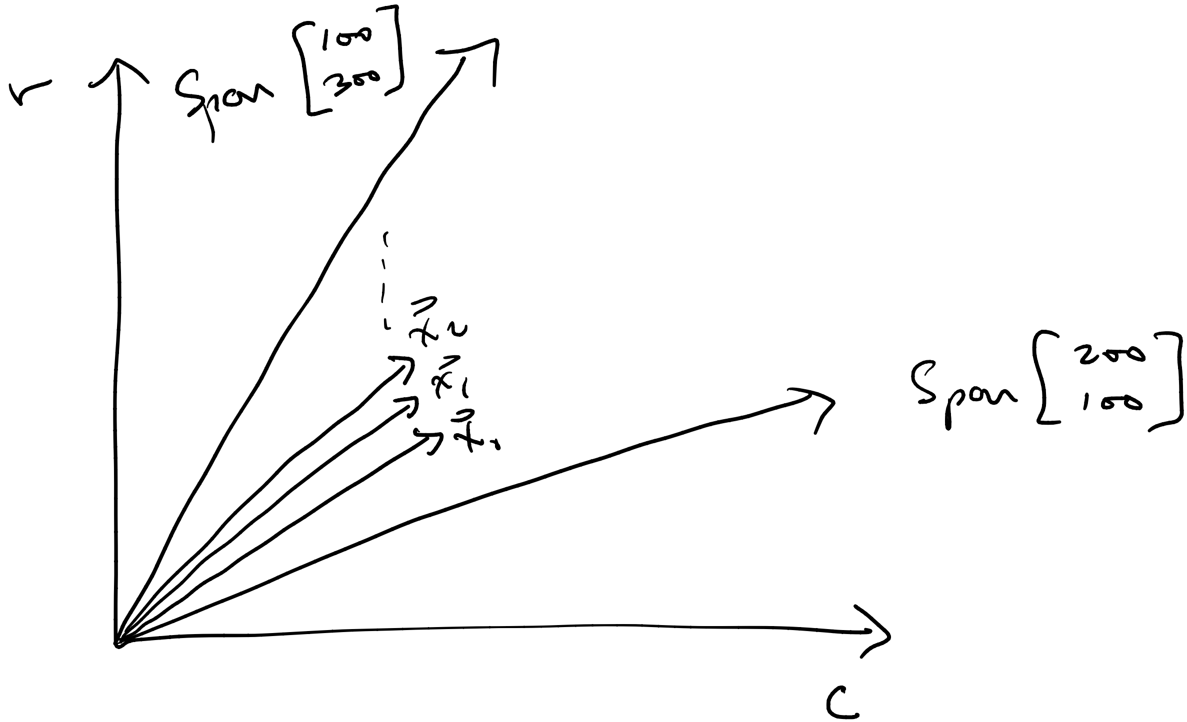
$$\Longrightarrow \begin{cases} c(t) = 200(1.1)^t + 800(0.9)^t \\ r(t) = 600(1.1)^t + 400(0.9)^t \end{cases}$$

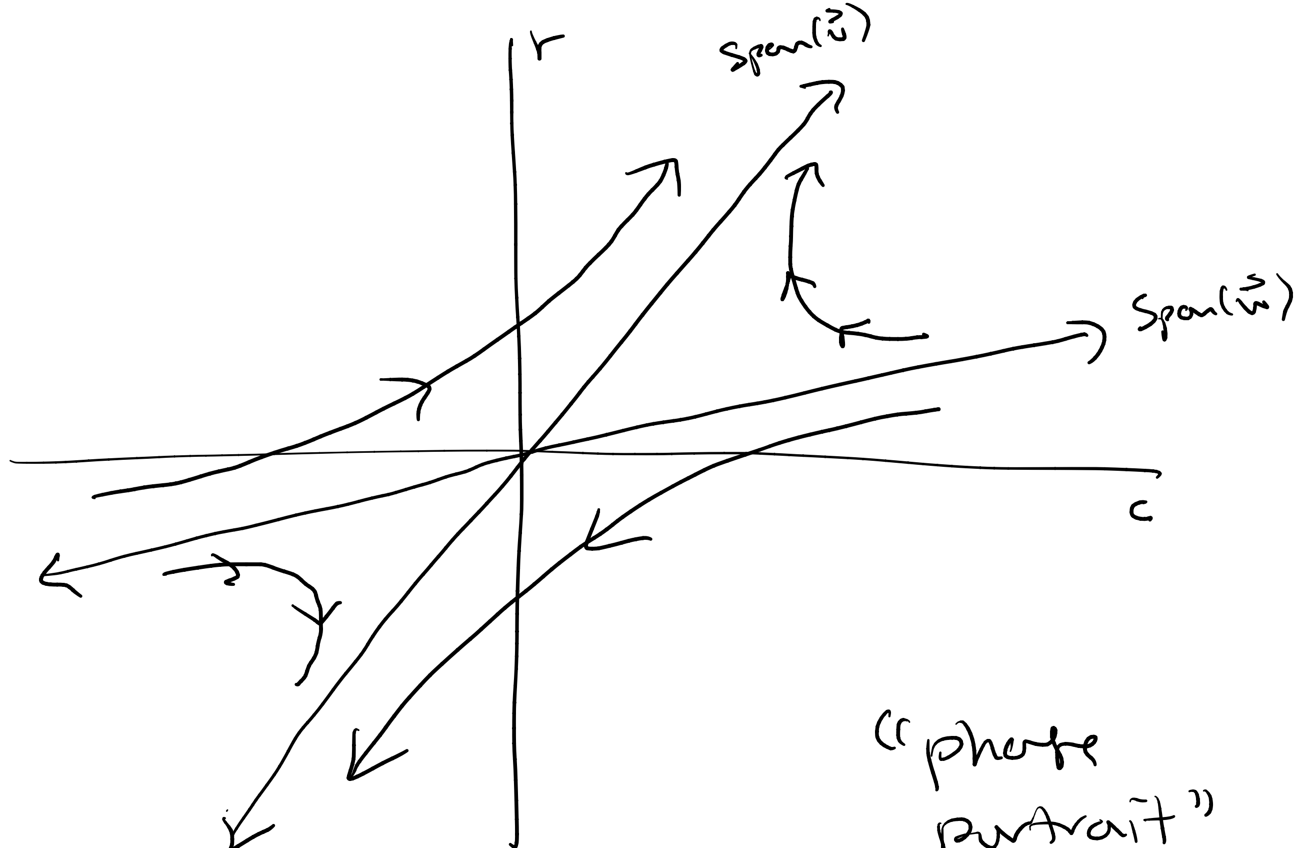
As $t \to \infty$, both approach exponential growth by 10%, and the ratio of rabbits to coyotes approaches

$$\lim_{t \to \infty} \frac{r(t)}{c(t)} = \lim_{t \to \infty} \frac{600(1.1)^t + 400(0.9)^t}{200(1.1)^t + 800(0.9)^t}$$

$$= \lim_{t \to \infty} \frac{600(1.1)^t}{200(1.1)^t}$$

$$= 3.$$


r
Span [100 300]
$\vec{x}_2$
$\vec{x}_1$
$\vec{x}_0$
Span [200 100]
c



"phase portrait"

The setup $\vec{x}_{t+1} = A\vec{x}_t$ is called a <u>discrete dynamical system</u> with <u>initial value</u> $\vec{x}_0$.

<u>Ex</u>

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[dl] \\ 3 \arrow[r] & 4 \arrow[u] \end{tikzcd}$$

mini-internet

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

transition matrix

This dynamical system models random web surfing. The relative influence of the pages is measured by the "equilibrium distribution", i.e., a distribution vector $\vec{x}_{equ}$ with

$$A\vec{x}_{equ} = \vec{x}_{equ}$$

We now recognize this as an eigenvector of A with eigenvalue $\lambda = 1$.