## Last time

- Discrete dynamical systems
- Exponential growth/decay, equilibria

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**Ex** Another transition matrix:

$$A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}$$

To understand this dynamical system, let's diagonalize. The eigenvalues of $A$ are $1/10$ of the eigenvalues of $\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}$:

$$\det\begin{bmatrix} 4-\lambda & 3 \\ 6 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda)-18$$

$$= \lambda^2 - 11\lambda + 10$$

$$= (\lambda-10)(\lambda-1)$$

So the eigenvalues of $A$ are $\lambda = 1, 1/10$.

$$E_1 = \ker \frac{1}{10}\begin{bmatrix} -6 & 3 \\ 6 & -3 \end{bmatrix} = \ker\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$E_{1/10} = \ker \frac{1}{10}\begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So $A$ is diagonalizable with

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix}.$$

Then $A^t = (SBS^{-1})^t$

$$S^{-1} = \frac{1}{3} S$$

$$= SB^tS^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/10^t \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1/10^t \\ 2 & -1/10^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 + 2/10^t & 1 - 1/10^t \\ 2 - 2/10^t & 2 + 1/10^t \end{bmatrix}$$

The long term behavior is given by

$$\lim_{t\to\infty} A^t = \lim_{t\to\infty} \frac{1}{3}\begin{bmatrix} 1 + 2/10^t & 1 - 1/10^t \\ 2 - 2/10^t & 2 + 1/10^t \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

So, for example, if $\vec{x}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, then

$$\vec{x}_t = \frac{1}{3}\begin{bmatrix} 1 + 2/10^t & 1 - 1/10^t \\ 2 - 2/10^t & 2 + 1/10^t \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 + 1/10^t \\ 4 - 1/10^t \end{bmatrix} \qquad \lim_{t\to\infty} \vec{x}_t = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

## DDS's by diagonalization $\vec{x}_t = A^t \vec{x}_0$

(1) Diagonalize $A$ as $S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

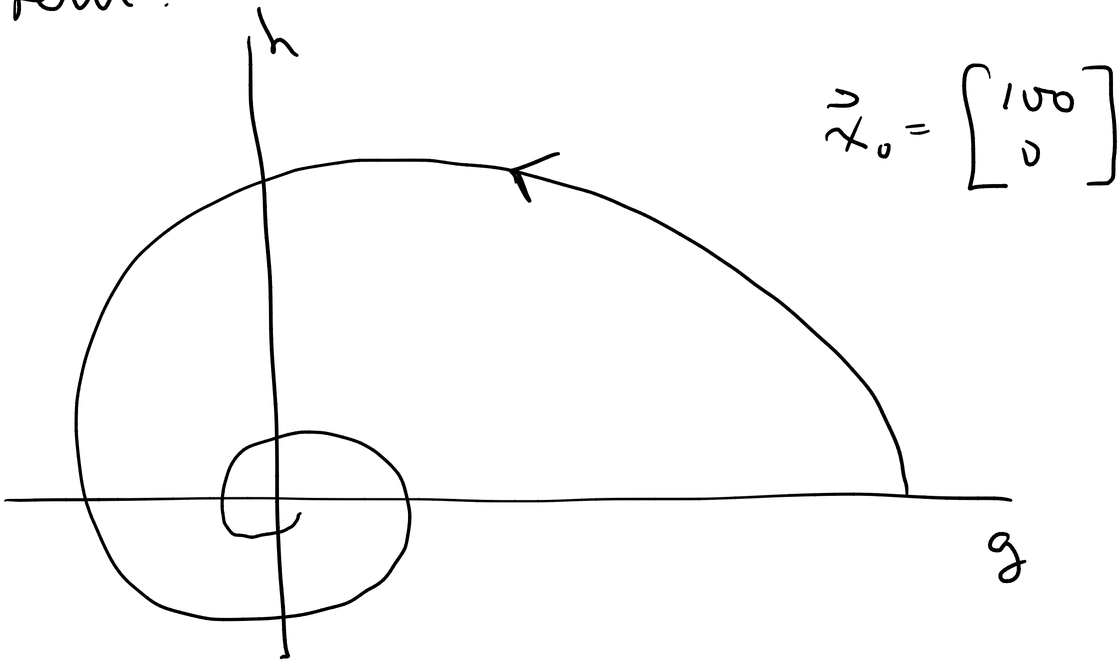
(2) $A^t = S \begin{bmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_n^t \end{bmatrix} S^{-1}$

<u>Ex</u> $A = \frac{1}{10}\begin{bmatrix} 9 & -4 \\ 1 & 9 \end{bmatrix}$ $\vec{x}_{t+1} = A\vec{x}_t$.

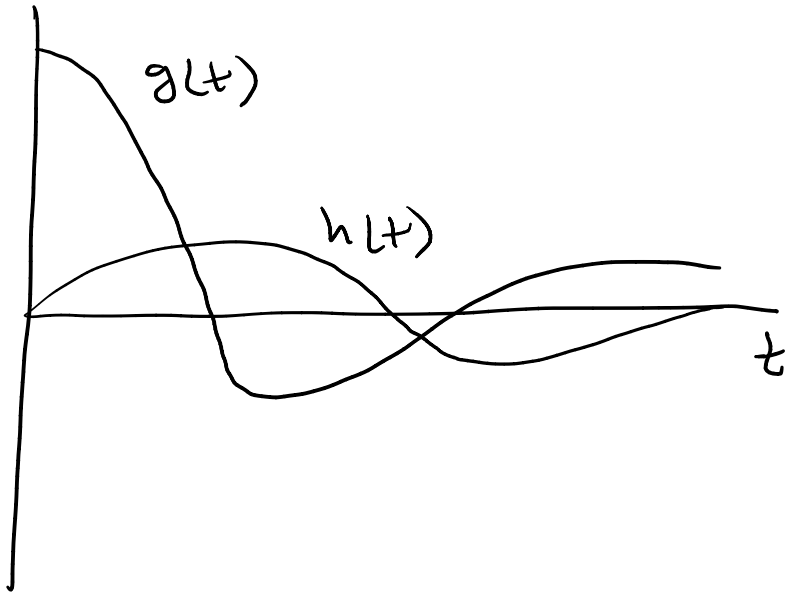
Again, the eigenvalues will be 1/10 of the roots of

$$\det\begin{bmatrix} 9-\lambda & -4 \\ 1 & 9-\lambda \end{bmatrix} = \lambda^2 - 18\lambda + 85$$

Since $b^2-4ac = 324 - 340 = -16 < 0$, there are no roots. Plotting values reveals a new, oscillatory behavior of the system:



Or, as functions of t:




To see how this new behavior is connected to the failure of our previous approach, we adjust our perspective: $f_A(\lambda) = \lambda^2 - 1.8\lambda + 0.85$ does have roots, so $A$ does have eigenvalues!

$$\lambda = \frac{1}{10}\,\frac{18 \pm \sqrt{-16}}{2}$$

$$= \frac{1}{10}(9 \pm 2i),$$

where $i$ is a symbol such that $i^2 = -1$. These are complex numbers.

## Complex numbers  $z = a + bi$, $a$ & $b$ real

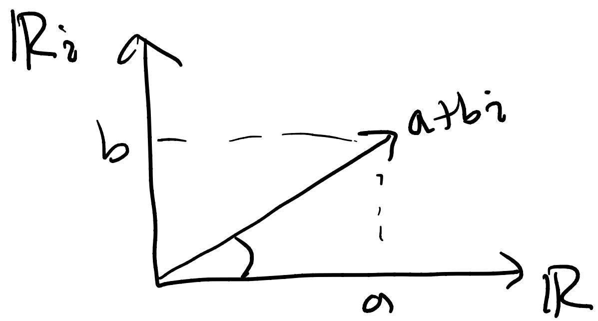
(0) $i^2 = -1$

(1) $a$ and $b$ are the real and imaginary parts, respectively

(2) $(a+bi) + (c+di) = a+c+(b+d)i$

(3) $(a+bi)(c+di) = ac-bd+(ad+bc)i$

(4) The complex plane $\mathbb{C}$:



(5) Polar form: $z = r(\cos\theta + i\sin\theta)$,

where $r = \sqrt{a^2+b^2}$

$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

(6) If $w = s(\cos\varphi + i\sin\varphi)$, then

$$zw = rs(\cos(\theta+\varphi) + i\sin(\theta+\varphi)).$$

(7) $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

(8) If $p(\lambda)$ is a polynomial with complex coefficients, then

$$p(\lambda) = k(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$$

Linear algebra works the same for matrices with complex entries, so we can diagonalize "over $\mathbb{C}$".

**Ex** $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$f_A(\lambda) = \lambda^2 + 1 = (\lambda + i)(\lambda - i), \quad \lambda = \pm i$$

$$E_i = \ker \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \ker \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} = \mathrm{Span} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$E_{-i} = \ker \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \ker \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} = \mathrm{Span} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$