Last time

$$A^t = SD^tS^{-1}$$

- DDS's by diagonalization
- Oscillation
- Complex numbers

We saw that the rotation matrix

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable over $\mathbb{C}$ with

$$S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

<u>Ex</u> More generally, the rotation-and-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is diagonalizable with

$$S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}.$$

Surprisingly, this tells us something strong about a general 2×2 matrix:

If A is a real 2×2 matrix with eigenvalues $a \pm bi$ $(b \neq 0)$ and $\vec{v} + i\vec{w}$ is an eigenvector with $\lambda = a+bi$, then

$$S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

So every such matrix is similar to a rotation and scaling matrix. Informally, complex eigenvalues capture oscillatory behavior.

$\underline{\text{Ex}}$ Returning to our example,

$$A = \frac{1}{10}\begin{bmatrix} 9 & -4 \\ 1 & 9 \end{bmatrix}, \quad \lambda = \frac{1}{10}(9 \pm 2i)$$

$$E_+ = \ker\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} = \ker\begin{bmatrix} 1 & -2i \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{w}}$$

$$S^{-1}AS = \frac{1}{10}\begin{bmatrix} 9 & -2 \\ 2 & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $\frac{1}{10}\begin{bmatrix} 9 & -2 \\ 2 & 9 \end{bmatrix}$ represents scaling by $\frac{1}{10}\sqrt{81+4} \approx 0.92$ and rotation by $\theta \approx 0.2$ radians, so

$S^{-1}A^tS = (S^{-1}AS)^t$ represents scaling by $\approx 0.92^t$ and rotation by $\approx 0.2t$ radians.

In particular, $\lim_{t\to\infty} A^t = 0$, and we have explained the "spiraling inward" picture from before.