Last time

- More on inverses
- Calculating A^{-1} $\quad [A \mid I_n] \xrightarrow{\text{row reduce}} [I_n \mid A^{-1}]$

Ex $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a general 2×2 matrix.

(0) Square ✓

(1) Case 1: $a \neq 0$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & -c/a & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\boxed{\text{If } ad-bc=0, \text{ not invertible!}}$$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/a + \frac{bc}{a(ad-bc)} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

Since $\frac{1}{a} + \frac{bc}{a(ad-bc)} = \frac{d}{ad-bc}$, we have

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Case 2</u> $a=0$. If $c=0$, then $rk(A)<2$, so A is not invertible. If $c \neq 0$,

$$\left[\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & b & 1 & 0 \end{array}\right]$$

Since $a=0$ and $c \neq 0$, $b=0$ if and only if $ad-bc=0$. If this happens, A is not invertible, as before. Otherwise,

$$\rightarrow \left[\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right]$$

Since
$$\begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

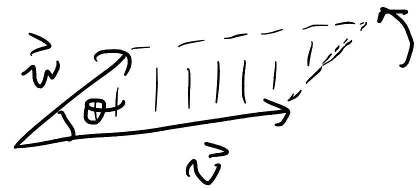
and $ad = 0$, we again have
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

So A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad-bc \neq 0$, in which case the inverse is

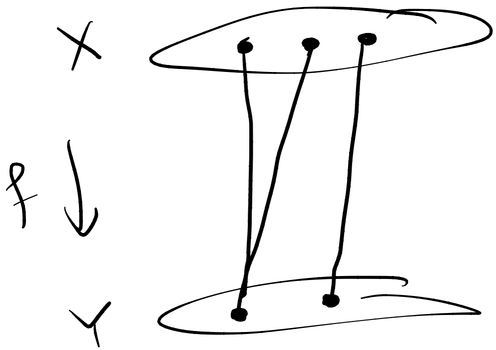
$$\boxed{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

The quantity $ad-bc$ is called the determinant of this matrix. We'll have more to say about determinants later on. For now, we'll interpret it geometrically:

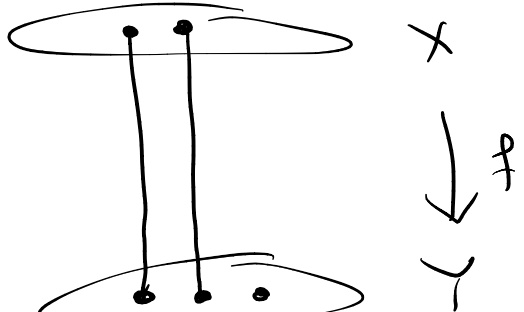
The magnitude of $\det([\vec{v}\ \vec{w}])$ is the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$. The sign is the sign of the angle $-\pi < \theta < \pi$ from $\vec{v}$ to $\vec{w}$



As we have seen, there are two ways in which invertibility can fail:



multiple solutions for some $\vec{b}$



no solution for some $\vec{b}$

We'd like to have a way to understand the degree of these failures.

For the second failure, this is easily done.

Def The *image* im(T) of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is the collection of all vectors $\vec{b}$ for which $T(\vec{x}) = \vec{b}$ has a solution.

Ex The image of projection onto a line L is the collection of vectors parallel to L.

Observation If im$(T) \neq \mathbb{R}^m$, then T is not invertible.

For the second failure, notice that, if $T(\vec{x}) = \vec{b}$ has infinitely many solutions, then so does $T(\vec{x}) = \vec{0}$, since

$$T(\vec{x}_1 - \vec{x}_2) = T(\vec{x}_1) - T(\vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

by linearity for any two solutions $\vec{x}_1$ and $\vec{x}_2$ of $T(\vec{x}) = \vec{b}$.

<u>Def</u> The <u>kernel</u> $\ker(T)$ of a linear transformation T is the collection of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

<u>Ex</u> The kernel of projection onto L is the collection of vectors perpendicular to the L

<u>Observation</u> If $\ker(T) \neq \{\vec{0}\}$, then T is not invertible.