Last time

- Inverse of general 2x2
- Image and kernel

These concepts relate to the matrix of T as follows:

$$\ker(T)=\{\vec{0}\} \iff rk(A)=n$$

$$\operatorname{im}(T)=\mathbb{R}^m \iff rk(A)=m.$$

So for an nxn matrix A:

$$\operatorname{im}(A)=\mathbb{R}^n \iff \begin{matrix}A \text{ is}\\ \text{invertible}\end{matrix} \iff \ker(A)=\{\vec{0}\}$$

If A isn't square, either could happen without the other:

<u>Ex</u>

$$\ker\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \left\{ t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \ker\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \{\vec{0}\}$$

$$\operatorname{im}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \mathbb{R}^2 \qquad \operatorname{im}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \mathbb{R}^2 \neq \mathbb{R}^3.$$

The kernel and image have properties in common.

(1) $\vec{0}$ is in both

(2) If $\vec{v}_1$ and $\vec{v}_2$ are in one, then so is $\vec{v}_1 + \vec{v}_2$.

(3) If $\vec{v}$ is in one, then so is $c\vec{v}$ for any number c.

A collection of vectors with these properties is called a (linear) <u>subspace</u>.

Ex $\{\vec{0}\}$ is a subspace

Ex A line through $\vec{0}$ is a subspace.

Ex A plane through $\vec{0}$ is a subspace

Ex $\operatorname{im}(T)$ is a subspace of $\mathbb{R}^m$ and $\ker(T)$ is a subspace of $\mathbb{R}^n$.

How can we understand what these spaces really are?

<u>Observation</u> Let $A = [\vec{v}_1 \cdots \vec{v}_n]$. A vector is in the image of A if and only if it is of the form

$$A\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n,$$

i.e., if it is a <u>linear combination</u> of the columns of A.

<u>Def</u> The <u>span</u> of a collection of vectors is the collection of all linear combinations of its vectors.

Ex The span of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ is $\mathbb{R}^2$.

Ex The span of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$ is the x-axis in $\mathbb{R}^2$.

So the image of A is the span of its column vectors. But this description can be very redundant.