Last time

- Row reduction algorithm / GJ elimination
- Number of solutions

0	inconsistent
1, ∞	consistent

We saw that the # of solutions is tied to the number and placement of pivots in the RREF.

Def (1) A variable whose column has no pivot in the RREF is called **free**.

(2) The <u>rank</u> of a matrix is the number of pivots in its RREF.

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \star \end{bmatrix} \end{array}$$

<u>Ex</u>

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ 2 free variable, rank 1

$$x = -5y - 3z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5r - 3s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

rank (coeff. matrix)	solution behavior
m (rows)	1 or ∞
$< n$	0 or ∞
n	0 or 1

Combining these, a square ($n \times n$) matrix is the coefficient matrix of a system w/ 1 solution iff it has rank n, i.e., the RREF is

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

<u>Observation</u> In this situation, there is a exactly one solution to

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

regardless of the values of $b_1, \ldots, b_m$.

Here's a contrasting example.

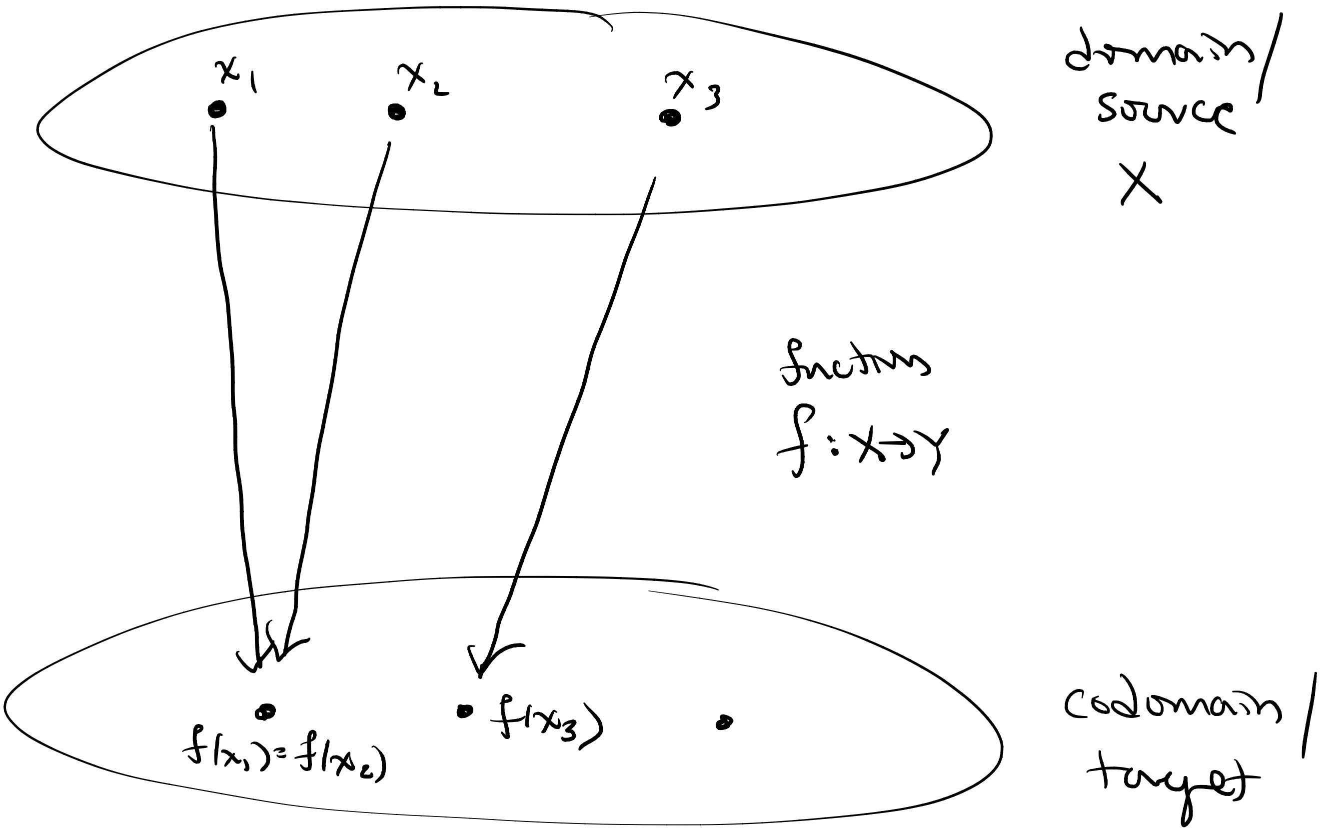
Ex The system

$$x + 0y = b_1 \qquad \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 0 & b_2 \end{array}\right]$$
$$0x + 0y = b_2$$

has a solution if and only if $b_2 = 0$.

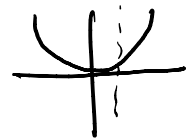
This difference is clarified by thinking of the system/matrix as a function.

A function is just a rule for assigning values to things.



Ex $f(x) = x^2$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

We write $f : \mathbb{R} \to \mathbb{R}$.



Ex $f(x) = x^2$ is a function from $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$. These are different functions because they have different targets

Ex Projection onto the x-axis is a function $P : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

The previous system of equations is the equation $P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, i.e., given a point in the target ($\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$), we are asking which points in the source are sent there by P.