Last time

- Free variables
- Rank
- Rank & solution behavior
- Functions

<u>Ex</u> Projection onto the x-axis is a function $P: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

The previous system of equations is the equation $P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, i.e., given a point in the target ($\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$), we are asking which points in the source are sent there by P.

<u>General construction</u> A matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$ determines a

function $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} =: A\vec{x}$$

This matrix product $A\vec{x}$ is the vector whose entries are the dot products of $\vec{x}$ with the rows of A.

So we can summarize the whole system of equations with the <u>matrix</u> <u>equation</u>

$$A\vec{x} = \vec{b}$$

We described the product $A\vec{x}$ above in terms of the <u>rows</u> of A. In terms of <u>columns</u>: the vector $A\vec{x}$ is obtained by scaling each column of A by the corresponding entry of $\vec{x}$ and adding the results, i.e.,

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \Rightarrow A\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n.$$

<u>Ex</u>

$$\begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 12 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 24 \end{bmatrix}.$$

An expression $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ is called a <u>linear combination</u> of the vectors $\vec{v}_1, \ldots, \vec{v}_n$. So:

$$\boxed{A\vec{x}=\vec{b} \text{ has a solution} \iff \vec{b} \text{ is a linear combination of the columns of } A}$$

The matrix product $A\vec{x}$ plays well with addition and scalar multiplication.

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} \quad \text{and} \quad A(c\vec{x}) = cA\vec{x}$$

OR

$$\boxed{A(a\vec{x}+b\vec{y}) = aA\vec{x} + bA\vec{y}}$$

<u>Def</u> A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called <u>linear</u> (or a linear transformation) if

$$T(a\vec{x}+b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

for every pair of vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$ and scalars a, b in $\mathbb{R}$.

<u>Big idea</u>

(1) The function determined by a matrix is a linear transformation.

(2) Every linear transformation is determined by a matrix.

Which matrix?

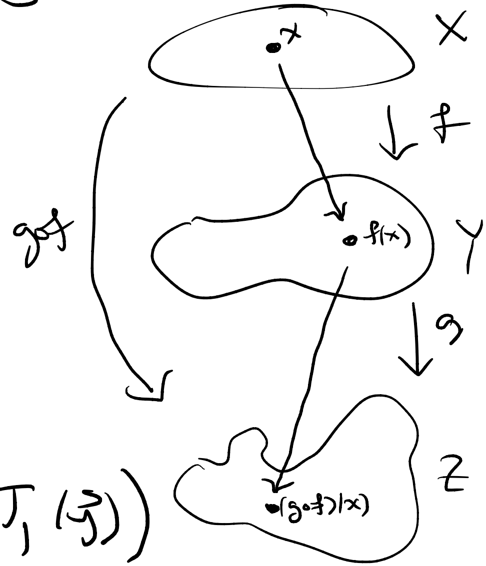
The $j^{\underline{th}}$ column of the matrix corresponding to the linear transformation T is the vector $T\vec{e}_j$, where

$$\vec{e}_j = \left[\begin{array}{c}0\\ \vdots\\ 1\\ \vdots\\ 0\end{array}\right]\begin{array}{l}\\ \\ \leftarrow j^{\text{th}}\\ \\ \\ \end{array},$$

i.e., it is the matrix

$$\begin{bmatrix} | & & | \\ T\vec{e}_1 & \cdots & T\vec{e}_n \\ | & & | \end{bmatrix}.$$

Given functions $f: X \to Y$ and $g: Y \to Z$, the __composite__ is the function $g \circ f : X \to Z$ defined by $(g\circ f)(x) = g(f(x))$. Applying this idea to linear transformations $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^\ell$, we have that



$$
\begin{aligned}
(T_2 \circ T_1)(a\vec{x} + b\vec{y}) &= T_2(T_1(a\vec{x} + b\vec{y})) \\
&= T_2(aT_1(\vec{x}) + bT_1(\vec{y})) \\
&= aT_2(T_1(\vec{x})) + bT_2(T_1(\vec{y})) \\
&= a(T_2 \circ T_1)(\vec{x}) + b(T_2 \circ T_1)(\vec{y}),
\end{aligned}
$$

so $T_2 \circ T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$ is also linear... so it corresponds to a matrix!