Last time

- Linearity $T(a\vec{x}+b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$
- Linear transformations $\longleftrightarrow$ matrices $\quad \begin{bmatrix} | & & | \\ T\vec{e}_1 & \cdots & T\vec{e}_n \\ | & & | \end{bmatrix}$
- Composite functions

$$(f \circ g)(x) = f(g(x))$$

We ended by observing that the composite of two linear transformations is also linear, so it corresponds to a matrix.

Def The **product** of an $m \times n$ matrix A and an $\ell \times m$ matrix B is the matrix BA associated to the linear transformation

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}^\ell \\ \vec{x} &\longmapsto B(A\vec{x}) \end{aligned}$$

$${}^{\ell}B^{m}\,{}^{m}A^{n}$$

Entrywise

The (i,j) entry of BA is the dot product of the $i^{\underline{th}}$ row of B w/ the $j^{\underline{th}}$ column of A

By columns

The $j^{\underline{th}}$ column of BA is the product of B with the $j^{\underline{th}}$ column of A.

Properties of matrix multiplication

- $A(BC) = (AB)C$ (associativity)
- $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$ (distributivity)
- $A(cB) = c(AB) = (cA)B$
- $AI_n = I_m A = A$

$$I_n = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n} \Big\} n$$

We continue to explore the idea of matrices acting on vectors by taking a second look at the example from the first lecture.

$$
\begin{tikzcd}
1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[dl] \\
3 \arrow[r] & 4 \arrow[u]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = A
$$

Before, we wanted to solve the equation

$$A\vec{x} = \vec{x},$$

which is to say find a vector that is left unchanged by the linear transformation determined by A. But what does this transformation <u>mean</u>?

<u>Observation</u> The j^{th} column contains the probabilities that a web surfer clicking randomly will end up on a given page.

So, if we input the vector $\vec{x} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$, "distribution vector"

representing a state where all internet users are evenly distributed, the result $A\vec{x}$ represents the distribution of surfers after clicking once randomly.

In this small example, we can solve the equation $A\vec{x} = \vec{x}$ by row reduction to obtain $\vec{x} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}$, but a realistically sized example would be impossible.

Instead, we let the surfers surf and watch what happens:

$$\vec{x} \xmapsto{\text{one click}} A\vec{x} \xmapsto{\text{two clicks}} A^2\vec{x} \mapsto \cdots \xmapsto{n \text{ clicks}} A^n\vec{x} \cdots$$

After 20 clicks, we have the following matrix, up to two significant figures:

$$A^{20} \approx \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 \\ 1/3 & 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

So inputting <u>any</u> initial distribution

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ with $x_1 + x_2 + x_3 + x_4 = 1$ gives

$$A^{20}\vec{x} \approx x_1 \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix} + x_2 \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix} + x_3 \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix} + x_4 \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$= (x_1 + x_2 + x_3 + x_4) \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

So the system can be solved (approximately) by calculating large powers of the transition matrix, which is cheaper than row reduction.