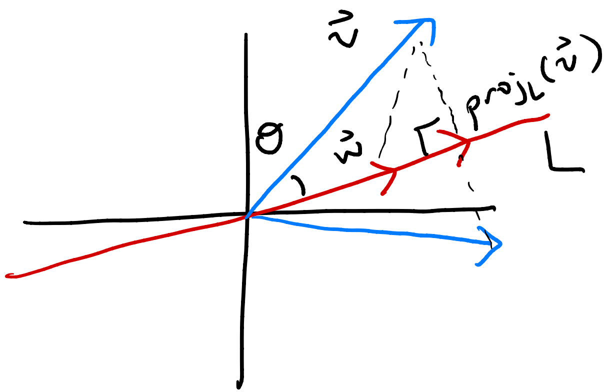
Last time

- Geometry of linear transformations

~~Scaling~~
~~projection~~
reflection
~~rotation~~
rotation
scaling
shear

Ex Reflection across a line L



$$\mathrm{ref}_L(\vec{v}) = \mathrm{proj}_L(\vec{v}) - \left(\vec{v} - \mathrm{proj}_L(\vec{v})\right)$$

$$= 2\,\mathrm{proj}_L(\vec{v}) - \vec{v}$$

For simplicity, replace $\vec{w}$ with a unit vector $\vec{u} = \dfrac{\vec{w}}{|\vec{w}|}$

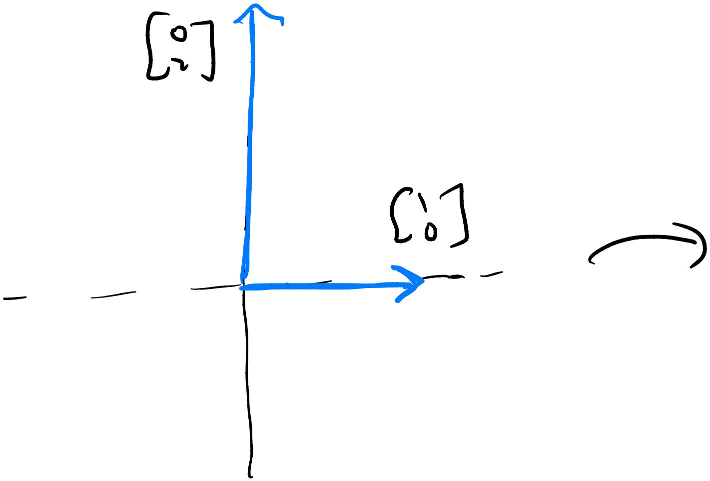
$$A = 2\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2u_1^2-1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2-1 \end{bmatrix}$$

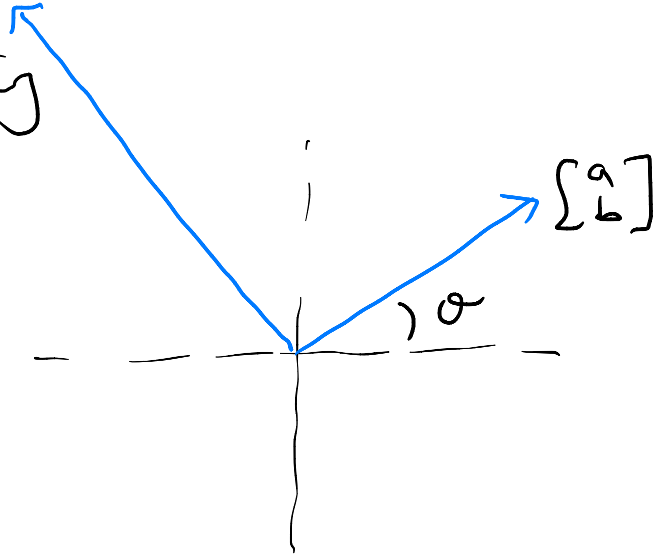
Since $u_1^2 + u_2^2 = 1$, this matrix is of the form

$$\boxed{\begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a^2+b^2=1}$$

and every such matrix is the matrix of a reflection.

<u>Ex</u> Rotation and scaling





Representing $\begin{bmatrix}a\\b\end{bmatrix}$ in polar coordinates, we're rotating by θ and scaling by r, where $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$, $r = \sqrt{a^2+b^2}$, so

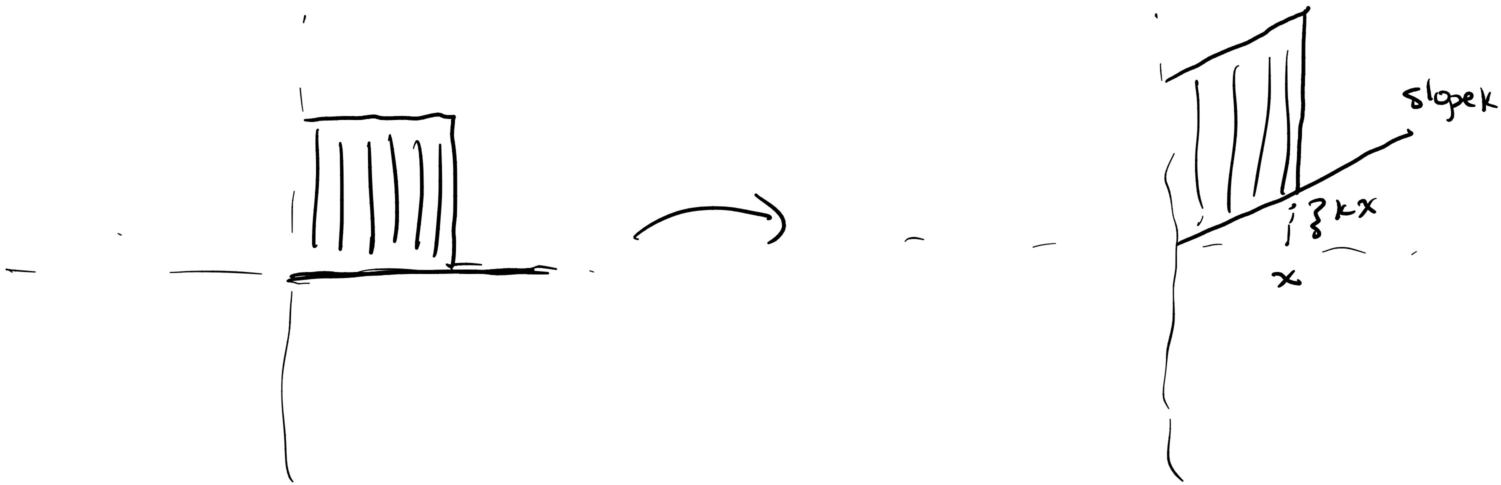
$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Ex Shears

For a vertical shear, we imagine placing a deck of cards on a ruler and raising the ruler:



If the slope of the raised ruler is k, the y coordinate of a vector is increased by kx, and the x coordinate is unchanged, i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y+kx \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Vertical Shear	Horizontal Shear
$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$	$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Moving on, we saw on the worksheet that the general solution to

$$A\vec{x} = \vec{b}, \quad A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

is

$$\vec{x} = \begin{bmatrix} b_1 - b_2 \\ -2b_1 + 3b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + b_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= B\vec{b}, \quad B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

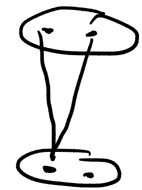
Similarly, the general solution to $B\vec{y} = \vec{c}$ is $\vec{y} = A\vec{c}$. So B is a matrix that "undoes" what A does, and vice versa.

Def Let $f: X \to Y$ be a function. An **inverse function** to f is a function $f^{-1}: Y \to X$ with $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = x$ for every x and y.

A function with an inverse is called **invertible**. We say a matrix is **invertible** if the corresponding linear transformation is so.

Invertibility is special!

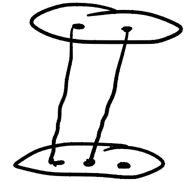
<u>Failure 1</u> There are $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. In this case, if f were invertible, we'd have



$$x_1 = f^{-1}(f(x_1)) = f^{-1}(f(x_2)) = x_2.$$

<u>Ex</u> If $A\vec{x} = \vec{b}$ has infinitely many solutions, then A is not invertible.

Failure 2 There is a y such that $f(x) \neq y$ for every x. In this case, if f were invertible, we'd have $f(f^{-1}(y)) = y$.



Ex $A\vec{x} = \vec{b}$ is inconsistent, then A is not invertible.

So:

A is invertible $\longleftrightarrow$ $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$