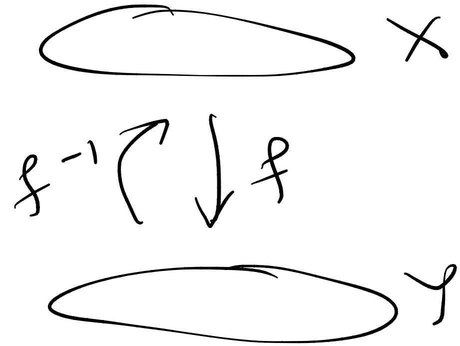
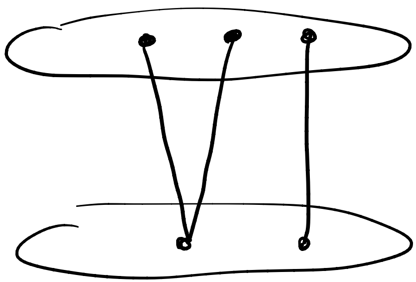
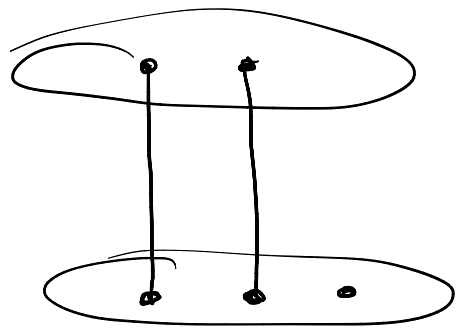
Last time

- More geometry
- Invertibility



Two failures:





A is invertible $\longleftrightarrow$ $A\vec{x} = \vec{b}$ has a unique solution for <u>every</u> $\vec{b}$

What does this say about RREF A?

(1) There is a pivot in every column, otherwise $A\vec{x}=\vec{0}$ has infinitely many solutions. So A has rank n.

(2) There is a pivot in every row, otherwise starting with $\left[\begin{array}{ccc|c} 1 & & & 0 \\ & 1 & & 0 \\ & & 0 & 1 \end{array}\right]$ and undoing row operations leads to an inconsistent system. So A has rank m.

$$\boxed{A \text{ is invertible} \iff A\vec{x}=\vec{b} \text{ has a unique solution for every } \vec{b} \iff A \text{ is } n\times n \text{ and has rank } n \iff \mathrm{RREF}(A)=I_n}$$

<u>Fact</u> If a linear transformation T is invertible, then T^{-1} is also linear.

Since linear transformations correspond to matrices and composition to products, we conclude:

A is invertible $\longleftrightarrow$ There is a matrix A^{-1} with $AA^{-1} = A^{-1}A = I_n$

<u>Fact</u> We only need to check that $AA^{-1} = I_n$ or $A^{-1}A = I_n$.

Characterizations of invertibility

- A is $n \times n$ and has rank n
- $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$
- $RREF(A) = I_n$
- there is a matrix A^{-1} such that $A^{-1}A = I_n = AA^{-1}$.

<u>Fact</u> A^{-1} is unique, and $(AB)^{-1} = B^{-1}A^{-1}$

Returning to our initial example, finding the inverse of $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ amounts to solving $A\vec{x} = \vec{b}$ for every $\vec{b}$. But

$$\vec{b} = b_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so if $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$A(b_1\vec{x}_1 + b_2\vec{x}_2) = b_1 A\vec{x}_1 + b_2 A\vec{x}_2 = \vec{b}.$$

So we only have to solve these two equations. We'd do this by row-reducing in each case, but in each case the steps are the same, so let's do them simultaneously:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

So $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

which means that

$$\left.\begin{aligned} A^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ A^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{aligned}\right\} \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

To find A^{-1}:

(0) If A isn't square, not invertible.

(1) If $\mathrm{RREF}([A \mid I_n]) = [I_n \mid B]$, then $B = A^{-1}$. Otherwise, not invertible.

(2) Check!