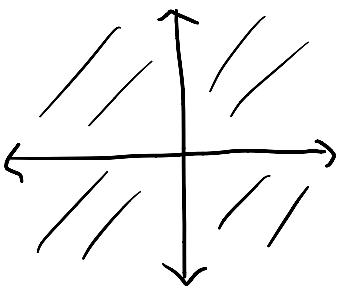
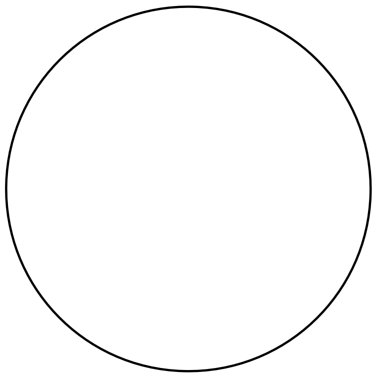
Q How to distinguish spaces?

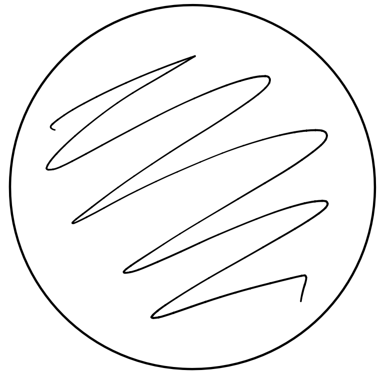
$\mathbb{R}^2$



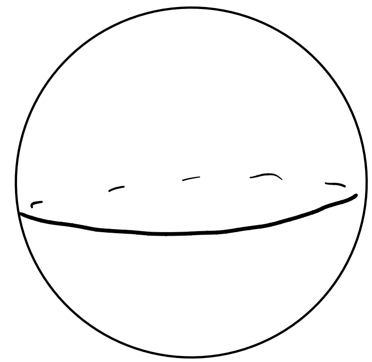
S^1

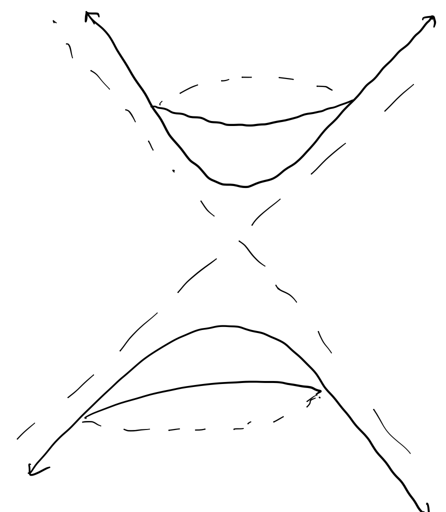


D^2

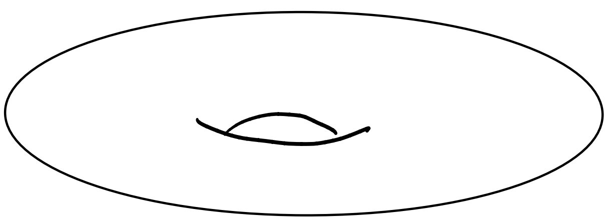


S^2

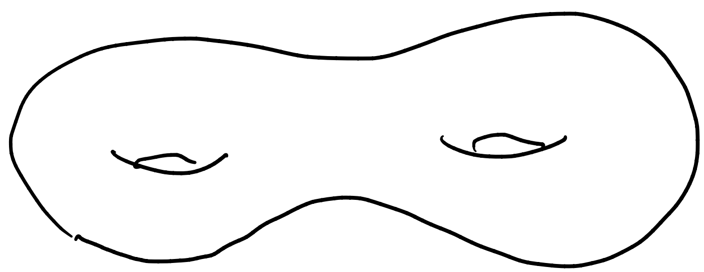




$T^2 = S^1 \times S^1$



Σ_2



$\mathbb{H}^2$

A1 (Point-set topology) Connectedness, compactness

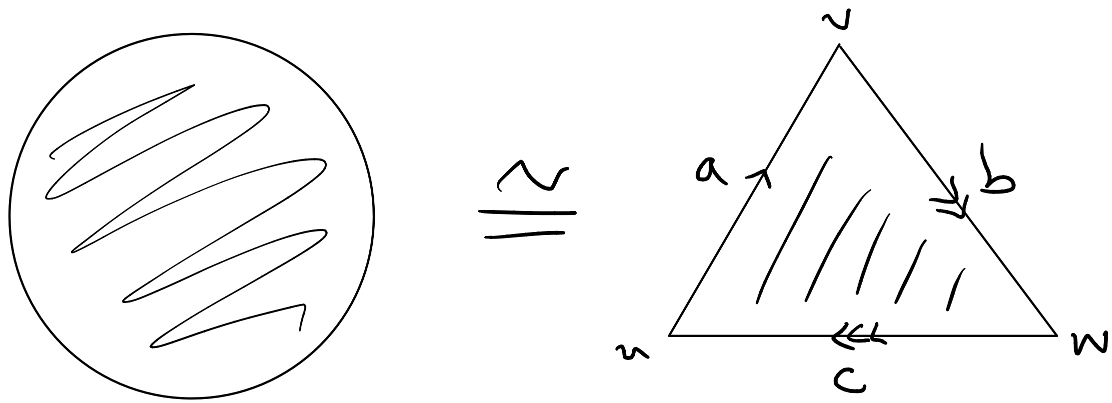
A2 (Algebraic topology) Count holes

Intuitively, it is clear that S^1 has a hole, while D^2 does not. Why? The obvious candidate hole (loop) is the boundary of something.

	2D data $\xrightarrow{\partial}$	1D data
D^2	{  } $\longmapsto$	$\{0\}$
S^1	$\emptyset \longmapsto$	$\{0\}$

The holes are what is left over

What if we describe the same situation with different combinatorics?



D^2 $\quad\quad$ Δ^2

$$\text{2D data} \xrightarrow{\partial_2} \text{1D data} \xrightarrow{\partial_1} \text{0D data}$$

	2D data	1D data	0D data
Δ^2	$\{\Delta\} \mapsto$	$\{a,b,c\} \mapsto$	$\{u,v,w\}$
$\partial\Delta^2$	$\emptyset \mapsto$	$\{a,b,c\} \mapsto$	$\{u,v,w\}$

In general, the boundary isn't a single datum but the "sum" of several. To make this idea sensible, we replace these sets with the corresponding free Abelian group:

	2D data $\xrightarrow{\partial_2}$	1D data $\xrightarrow{\partial_1}$	0D data
Δ^2	$\mathbb{Z}\langle \blacktriangle \rangle \longrightarrow$	$\mathbb{Z}\langle a, b, c\rangle \longrightarrow$	$\mathbb{Z}\langle u, v, w\rangle$
$\partial\Delta^2$	$0 \longrightarrow$	$\mathbb{Z}\langle a, b, c\rangle \longrightarrow$	$\mathbb{Z}\langle u, v, w\rangle$

$\partial \blacktriangle = a+b+c \qquad \partial a = v-u$ etc.

<u>Observation</u> A collection of edges forms a loop if and only if the endpoints are paired, so the "candidate holes" are $\ker \partial_1$.

<u>Observation</u> The "filled in holes" are $\operatorname{im} \partial_2$, so "what's left over" is the quotient

$$H_1 := \ker \partial_1 / \operatorname{im} \partial_2$$

<u>Exercise</u>

	$\operatorname{im} \partial_2$	$\ker \partial_1$	H_1
Δ^2	$\langle a+b+c \rangle$	$\langle a+b+c \rangle$	0
$\partial \Delta^2$	0	$\langle a+b+c \rangle$	$\mathbb{Z}$

This formula suggests a conceptual generalization.

$$\boxed{\begin{array}{c}\text{combinatorial}\\ \text{description of } X\end{array}}$$

$$\Downarrow$$

$$\boxed{\cdots \longrightarrow \begin{array}{c}\text{3D}\\ \text{data}\end{array} \xrightarrow{\partial_3} \begin{array}{c}\text{2D}\\ \text{data}\end{array} \xrightarrow{\partial_2} \begin{array}{c}\text{1D}\\ \text{data}\end{array} \xrightarrow{\partial_1} \begin{array}{c}\text{0D}\\ \text{data}\end{array} \xrightarrow{\partial_0} \begin{array}{c}\text{-1D}\\ \text{data}\end{array} = 0}$$

$$\Downarrow$$

$$\boxed{H_n(X) := \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}}$$

For this to make sense, we only need

$$\operatorname{im}(\partial_{n+1}) \subseteq \ker(\partial_n) \iff \partial_n \circ \partial_{n+1} = 0.$$

Def A chain complex is a sequence $\{C_n\}_{n\in\mathbb{Z}}$ of Abelian groups, together with homomorphisms

$$\partial_n : C_n \longrightarrow C_{n-1}$$

such that $\partial_n \circ \partial_{n+1} = 0$.

Rmk (1) This data can be collected in a "graded Abelian group" $C = \bigoplus_{n\in\mathbb{Z}} C_n$, together with an endomorphism $\partial : C \to C$. Being a chain complex is then the pair of conditions $\partial(C_n) \subseteq C_{n-1}$ and $\partial^2 = 0$. For this reason, we often write (C, ∂) or just C.

(2) The map ∂ is often called the <u>differential</u> or <u>boundary operator</u>. Other common letters are d and δ.

<u>Def</u> Let (C, ∂) be a chain complex.

The group of n-<u>cycles</u> is

$$Z_n := Z_n(C) := \ker(\partial_n).$$

The group of n-<u>boundaries</u> is

$$B_n := B_n(C) := \operatorname{im}(\partial_{n+1}).$$

The n^{th} <u>homology</u> group of C is

$$H_n(C) = Z_n / B_n.$$