Last time

- Degree
- Classical applications

One could go on: the Perron-Frobenius theorem, the fundamental theorem of algebra, the Ham Sandwich theorem, the Borsuk-Ulam theorem...

Thm (Generalized Jordan curve theorem)
If $f: S^r \to S^m$ is an embedding, then

$$\tilde{H}_n(S^m \setminus \mathrm{im}(f)) \cong \begin{cases} \mathbb{Z}, & n = m-r-1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma If $f: D^r \to S^m$ is an embedding, then $\tilde{H}_*(S^m \setminus f(D^r)) = 0$.

Assuming this result for now, we prove the theorem.

Proof of thm We proceed by induction on r. When $r=0$, $S^m \setminus f(S^0) \cong \mathbb{R}^m \setminus \{0\} \simeq S^{m-1}$, and the claim holds. For the induction step, write $S^r = D_+ \cup D_-$ with $D_\pm \cong D^r$, $D_+ \cap D_- = S^{r-1}$, and set

$$U_\pm = S^m \setminus f(D_\pm).$$

Then we have

$$\mathcal{U}_+ \cap \mathcal{U}_- = S^m \setminus f(S^r)$$

$$\mathcal{U}_+ \cup \mathcal{U}_- = S^m \setminus f(S^{r-1})$$

and the Mayer-Vietoris sequence is

$$\cancelto{0}{\tilde{H}_{n+1}(\mathcal{U}_+)} \oplus \cancelto{0}{\tilde{H}_{n+1}(\mathcal{U}_-)} \longrightarrow \tilde{H}_{n+1}(\mathcal{U}_+ \cup \mathcal{U}_-) \xrightarrow{\cong} \tilde{H}_n(\mathcal{U}_+ \cap \mathcal{U}_-) \longrightarrow \cancelto{0}{\tilde{H}_n(\mathcal{U}_+)} \oplus \cancelto{0}{\tilde{H}_n(\mathcal{U}_-)}$$

The outer terms vanish by Lemma 1, so the claim follows from the induction hypothesis.

$\square$

Before proving Lemma 1, we draw a few consequences.

Cor If $r > n$, then S^r does not embed in S^n.

Cor If $f: S^{m-1} \to S^m$ is an embedding, then $S^m \setminus \mathrm{im}(f)$ has two path components, each acyclic (i.e., having the homology of a point).

Cor If $f: S^{m-1} \hookrightarrow \mathbb{R}^m$ is an embedding w/ $m \geq 2$ $\mathbb{R}^m \setminus \mathrm{im}(f)$ has two path components, one bounded and acyclic and one unbounded with the homology of S^{m-1}.

Cor (Invariance of domain) If $U \subseteq \mathbb{R}^m$ is open and $g : U \hookrightarrow \mathbb{R}^m$ is an embedding, then $\operatorname{im}(g)$ is open in $\mathbb{R}^m$.

Proof Given $x \in U$ and ε s.t. $\overline{B_\varepsilon(x)} \subseteq U$, $g(\partial B_\varepsilon(x))$ separates $S^m \supseteq \mathbb{R}^m$ into two path components, which are $S^m \setminus g(\overline{B_\varepsilon(x)})$ and $g(B_\varepsilon(x))$, since these subsets are disjoint, path connected and cover.
These path components are also the connected components, hence open, since there are finitely many of them. $\square$

Cor (Invariance of dimension) Let U and V be nonempty subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If $U \cong V$, then $m=n$.

Proof WLOG $m<n$, and consider the composite

$$V \cong U \subseteq \mathbb{R}^m \xhookrightarrow{i} \mathbb{R}^n,$$

where i is the standard embedding. As a composite of embeddings, this map is an embedding, so its image is open by invariance of domain. But $i(\mathbb{R}^m)$ contains no open ball, hence no open set. $\square$

We return to Lemma 1.

<u>Lemma 2</u> Let $i_k : U_k \subseteq X$ be open for $k \geq 0$, and suppose that

(1) $X = \bigcup_{k \geq 0} U_k$, and

(2) for $k \leq l$, $i_{kl} : U_k \subseteq U_l$.

Given $\alpha \in H_*(U_k)$, if $(i_k)_* \alpha = 0$, then $(i_{kl})_* \alpha = 0$ for some $l \geq k$.

<u>Proof</u> Writing $\alpha = [c]$, $(i_k)_\# c = \partial c'$ by assumption. If $c' = \sum_{i=1}^m n_i \sigma_i$, then the space $\bigcup_{i=1}^m \mathrm{im}(\sigma_i)$ is compact, so lies in U_l

for some $l \geq n$. Thus, there exists $c'' \in C_*(U_l)$ such that $(i_l)_\# c'' = c'$. We have

$$
\begin{aligned}
(i_l)_\# \partial c'' &= \partial (i_l)_\# c'' \\
&= \partial c' \\
&= (i_n)_\# c \\
&= (i_l)_\# (i_{nl})_\# c
\end{aligned}
$$

$$\Longrightarrow \partial c'' = (i_{nl})_\# c \Longrightarrow (i_{nl})_* \alpha = 0. \qquad \square$$

We now prove Lemma 1.

<u>Proof</u> We proceed by induction on r. If $r=0$, then $S^m \setminus f(D^0) \cong \mathbb{R}^m$, and the claim holds. For the induction step, suppose α is a nonzero homology class in $S^m \setminus f(D^r)$. In light of the homeomorphism $D^r \cong D^{r-1} \times [0,1]$, write

$$U = S^m \setminus f(D^{r-1} \times [0, 1/2])$$

$$V = S^m \setminus f(D^{r-1} \times [1/2, 1]).$$

Then $U \cap V$ is the space of interest, and the induction hypothesis applies to

$$U \cup V = S^m \setminus f(D^{r-1} \times \{1/2\}),$$

so the Mayer-Vietoris sequence is

$$\cancelto{0}{\tilde{H}_{n+1}(U \cup V)} \to \tilde{H}_n(S^m \setminus f(D^r)) \xrightarrow{\cong} \tilde{H}_n(U) \oplus \tilde{H}_n(V) \to \cancelto{0}{\tilde{H}_n(U \cup V)}.$$

Without loss of generality, $\alpha \mapsto \alpha_1 \in \tilde{H}_n(U)$ with $\alpha_1 \neq 0$. Repeating this argument, we obtain a sequence of closed intervals $[0,1] \supseteq [0,1/2] = I_1 \supseteq I_2 \supseteq \cdots$ and classes

$$\begin{array}{ccc} \alpha & \longmapsto & \alpha_k \neq 0 \\ \tilde{H}_n(S^m \setminus m(f)) & \longrightarrow & \tilde{H}_n(S^m \setminus f(D^{r-1} \times I_k)), \end{array}$$

where the length of I_k is $1/2^k$. Then $\bigcap_{k\geq 0} I_k = \{a\}$ for some $a \in [0,1]$. Applying Lemma 1 to the open cover $\{S^m \setminus f(D^{r-1} \times I_k)\}$, it follows that the image of α is nonzero in $\tilde{H}_n(S^m \setminus f(D^{r-1} \times \{x\})) = 0$, a contradiction.

$\square$