Last time

- Informal discussion of "holes", homology
- Chain complexes

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

$$\partial^2 = 0$$

Idea

$\boxed{\text{Combinatorial description of } X}$

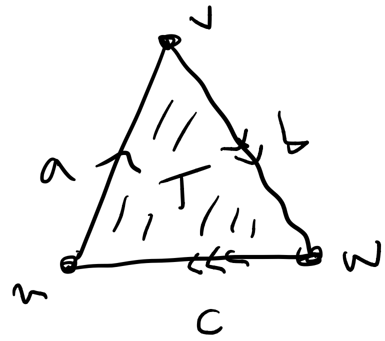
$\Downarrow$

$\boxed{\text{chain complex}}$

$\Downarrow$ ✓

$\boxed{\text{homology groups}}$

Ex



$$0 \to \mathbb{Z}\langle T\rangle \to \mathbb{Z}\langle a,b,c\rangle \to \mathbb{Z}\langle u,v,w\rangle \to 0$$

$$T \mapsto a+b+c$$

$$a \mapsto v-u$$

$$b \mapsto w-v$$

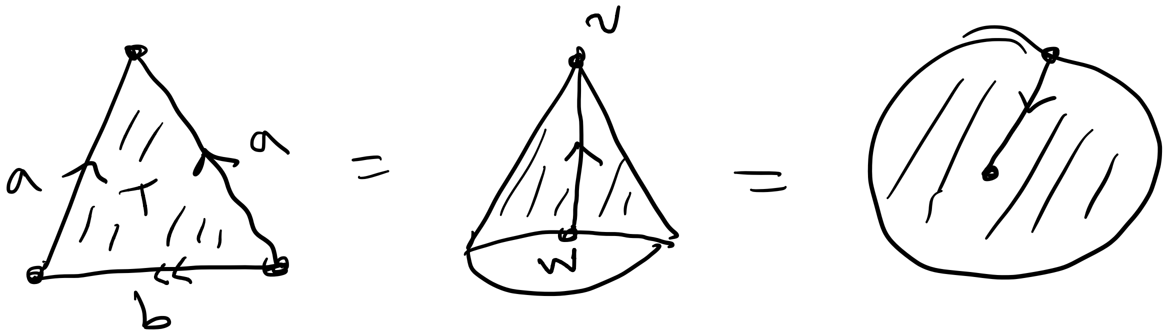
$$c \mapsto u-w$$

$$H_0 = \frac{\mathbb{Z}\langle u,v,w\rangle}{\langle v-u, w-v, u-w\rangle} \cong \mathbb{Z}$$

$$H_1 = \frac{\mathbb{Z}\langle a+b+c\rangle}{\mathbb{Z}\langle a+b+c\rangle} = 0$$

$$H_2 = 0$$

Ex



$\cong D^2$

$$0 \to \mathbb{Z}\langle T\rangle \to \mathbb{Z}\langle a,b\rangle \to \mathbb{Z}\langle v,w\rangle \to 0$$

$$T \longmapsto a - a + b = b$$

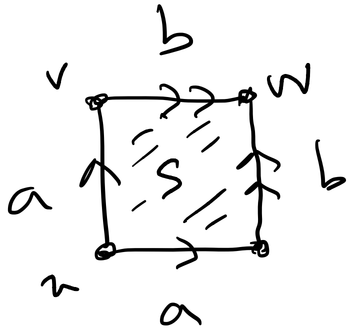
$$a \longmapsto v - w$$

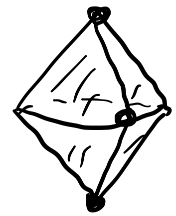
$$b \longmapsto 0$$

$$H_0 = \frac{\mathbb{Z}\langle v,w\rangle}{\langle v-w\rangle} \qquad H_1 = \frac{\mathbb{Z}\langle b\rangle}{\mathbb{Z}\langle b\rangle} = 0 \qquad H_2 = 0$$

$\cong \mathbb{Z}$

The complexes here and for $\Delta^2 \cong D^2$ differ, but the homology groups agree (to be continued).

Ex 

$=$  $\cong S^2$

$$0 \longrightarrow \mathbb{Z}\langle s \rangle \longrightarrow \mathbb{Z}\langle a,b \rangle \longrightarrow \mathbb{Z}\langle u,v,w \rangle \longrightarrow 0$$

$$\longmapsto a+b-b-a=0$$

$$a \longmapsto v-u$$

$$b \longmapsto w-v$$

$H_0 \cong \mathbb{Z}$ as before

$H_1 = 0$

$H_2 \cong \mathbb{Z}$

We imagine that $H_2 \neq 0$ reflects the fact that S^2 encloses space, unlike D^2.

<u>Idea</u> The n^{th} homology group of a space X (once we've defined it!) should see the "n-dimensional holes" in X.

It remains to make precise the first step of the recipe:

$$\boxed{\begin{array}{c}\text{Combinatorial} \\ \text{description of } X\end{array}}$$

$$\Downarrow$$

$$\boxed{\text{chain complex}}$$

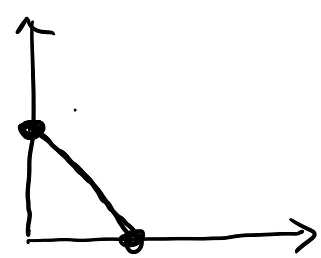
As we will see, there are many (equivalent) ways to do this. We begin with the higher dimensional analogues of triangles.

<u>Def</u> The <u>standard n-simplex</u> is the subspace

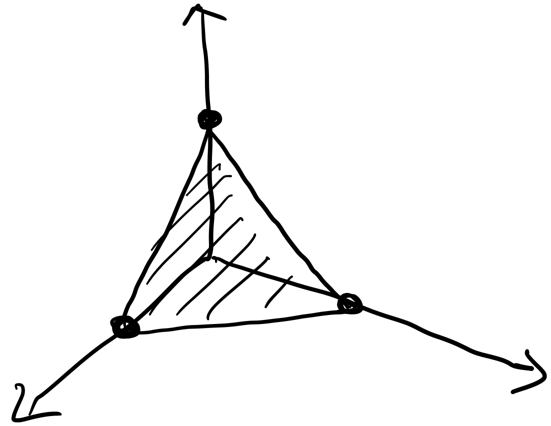
$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \right\}.$$

<u>Ex</u> Δ^0 is a singleton

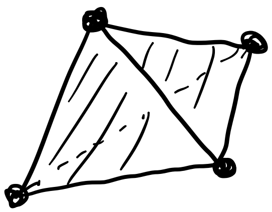
<u>Ex</u> Δ^1 is a segment



Ex Δ^2 is a triangle



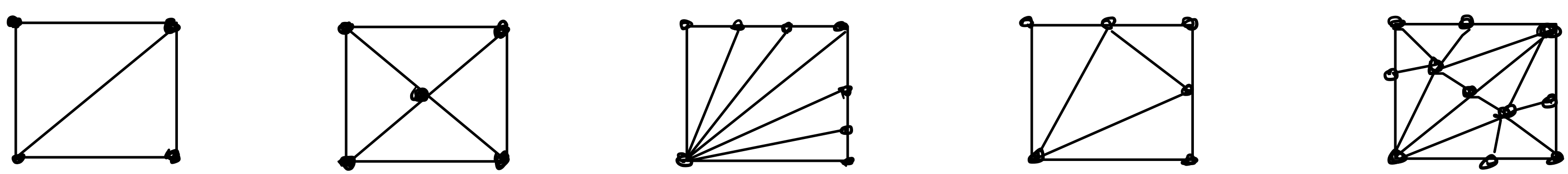
Ex Δ^3 is a tetrahedron



The boundary of Δ^n is the union of $n+1$ faces, each homeomorphic to Δ^{n-1}:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook, "t_i = 0"] & \mathbb{R}^{n+1} \\
\Delta^{n-1} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, dashed, "\eta_i"] & \Delta^n \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}.$$

One can easily imagine generalizing the idea of a triangulation to encompass methods of gluing simplices along faces to form topological spaces of arbitrary dimension. The resulting theories (simplicial complexes, Δ-complexes, simplicial and semisimplicial sets) all give good theories of homology. But a problem inevitably arises, namely that different combinatorics produce different chain complexes for the same space:



Experiment suggests that the homology is independent of choice. But why?

Observation Every n-simplex in a decomposition of a space is in particular a continuous map from Δ^n.

Def A singular n-simplex of a space X is a map $\sigma : \Delta^n \to X$. The group of singular n-chains of X is the free Abelian group $C_n(X)$ generated by the set of all singular n-simplices of X.

Warning $C_n(X)$ is usually very large (e.g., uncountably generated), and even simple spaces have singular simplices of all dimensions.

__Ex__ For any X, $C_0(X) \cong \mathbb{Z}\langle X \rangle$

__Ex__ If X is a singleton, then $C_n(X) \cong \mathbb{Z}$ for every $n \geq 0$.

Define $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \, \sigma \circ \eta_i$$

$$\begin{array}{c} \Delta^{n-1} \\ \downarrow \eta_i \\ \Delta^n \\ \downarrow \sigma \\ X \end{array}$$

We'd like to say that ∂ makes $C_*(X) := \bigoplus_{n \geq 0} C_n(X)$ into a chain complex.

__Lemma 1__ $\partial_n \circ \partial_{n+1} = 0$

Lemma 2 If $i \leq j$, then $\eta_i \circ \eta_j = \eta_{j+1} \circ \eta_i$.

Proof Both are induced by the map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^{n+2}$$

$$(t_0, \dots, t_{n-1}) \longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1})$$

$\square$

Proof of Lemma 1

$$\partial_n(\partial_{n+1}\sigma) = \partial_n\left(\sum_{i=0}^{n+1}(-1)^i \sigma \circ \eta_i\right)$$

$$= \sum_{i=0}^{n+1}\sum_{j=0}^{n}(-1)^{i+j}\sigma \circ \eta_i \circ \eta_j$$

$$= \sum_{0 \leq i \leq j \leq n}(\text{— '' —}) + \sum_{0 \leq j < i \leq n+1}(\text{— '' —})$$

$$= \sum_{0\le i\le j\le n} (-1)^{i+j}\, \sigma\circ\eta_{j+1}\circ\eta_i + \sum_{0\le j< i\le n+1} \left(\text{---}\ ''\ \text{---}\right)$$

$$= \sum_{0\le i< j\le n+1} (-1)^{i+j-1}\, \sigma\circ\eta_{j}\circ\eta_i + \sum_{0\le j< i\le n+1} (-1)^{i+j}\, \sigma\circ\eta_{i}\circ\eta_j$$

$$= \sum_{0\le i< j\le n+1} \left[(-1)^{i+j-1} + (-1)^{i+j}\right] \sigma\circ\eta_{j}\circ\eta_i$$

$$= 0. \qquad \square$$