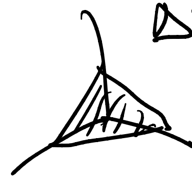
Last time



Δ^2

- Examples
- Standard simplices
- Singular simplices, $C_*(X)$ $\quad \Delta^n \hookrightarrow X$
- $\partial^2 = 0$

Def The chain complex $(C_*(X), \partial)$ is called the complex of singular chains in X. Its homology is the (singular) homology of X, denoted $H_*(X)$.

Warning Calculating $H_*(X)$ from this definition is almost always impossible.

<u>Ex</u> ($X = pt$)

$$C_*(pt) = \left(\to \mathbb{Z}\langle \sigma_n \rangle \to \mathbb{Z}\langle \sigma_{n-1} \rangle \to \cdots \to \mathbb{Z}\langle \sigma_0 \rangle \to 0 \right),$$

where $\sigma_n : \Delta^n \to pt$ is the unique map.

$$\begin{tikzcd}
\Delta^{n-1} \arrow[d, "\eta_i"] \arrow[dd, bend right=60, "\sigma_{n-1}"'] \\
\Delta^n \arrow[d, "\sigma_n"] \\
pt
\end{tikzcd}$$

If $n > 0$, then

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n \circ \eta_i$$

$$= \sum_{i=0}^{n} (-1)^i \sigma_{n-1}$$

$$= \begin{cases} \sigma_{n-1}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

$$\ker(\partial_n) = \begin{cases} C_n(pt), & n \text{ odd or } n=0 \\ 0, & n>0 \text{ even} \end{cases}$$

$\Rightarrow$

$$\operatorname{im}(\partial_{n+1}) = \begin{cases} C_n(pt), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\Rightarrow \quad H_n(pt) \cong \begin{cases} \mathbb{Z}, & n=0 \\ 0, & n>0. \end{cases}$$

At this point, it is very unclear that singular homology is

(1) useful
(2) meaningful
(3) calculable

Toward the first objection, we make the following.

<u>Observation</u> Given $f: X \to Y$ and a singular n-simplex σ of X, $f \circ \sigma$ is a singular n-simplex of Y.

$$\begin{tikzcd} \Delta^n \arrow[d, "\sigma"] \\ X \arrow[d, "f"] \\ Y \end{tikzcd}$$

We obtain a homomorphism

$$C_*(f) : C_*(X) \longrightarrow C_*(Y)$$
$$\sigma \longmapsto f \circ \sigma.$$

What more do we need?

Suppose C and D are chain complexes and a homomorphism $\varphi : C \to D$ s.t. $\varphi(C_n) \subseteq D_n$.

$$\begin{array}{ccc}
C_n & \xrightarrow{\varphi} & D_n \\
\cup\!| & & \cup\!| \\
Z_n(C) & \overset{?}{\dashrightarrow} & Z_n(D) \\
\downarrow & & \downarrow \\
H_n(C) & \overset{??}{\dashrightarrow} & H_n(D)
\end{array}$$

Def We say φ is a **chain map** iff $\partial \circ \varphi = \varphi \circ \partial$.

$$\begin{array}{ccc}
C_n & \xrightarrow{\varphi} & D_n \\
\partial \downarrow & & \downarrow \partial \\
C_{n-1} & \xrightarrow{\varphi} & D_{n-1}
\end{array}$$

Exercise A chain map respects cycles and boundaries, so induces a homomorphism on homology.

Def A quasi-isomorphism is a chain map inducing an isomorphism on homology.

Exercise $C_*(f)$ is a chain map.

So a map $f: X \to Y$ induces a homomorphism

$$f_*: H_*(X) \to H_*(Y).$$

Exercise $(f \circ g)_* = f_* \circ g_*$ and $(id_X)_* = id_{H_*(X)}$.

This extra structure provides a means of comparing the homology of different spaces. It also provides an answer to objection (2) above.

Def A space X is <u>contractible</u> if there is a map $H: X \times [0,1] \to X$ such that

$$H(x,t) = \begin{cases} x & \text{if } t=1 \\ x_0 & \text{if } t=0 \end{cases}$$

for some $x_0 \in X$.

Thm If X is contractible, then the unique map $X \to pt$ induces an isomorphism on homology. In particular, $H_n(X) = 0$ for $n>0$.

Proof Fix H and x_0 as above, and let $f: X \to pt$ be the unique map and

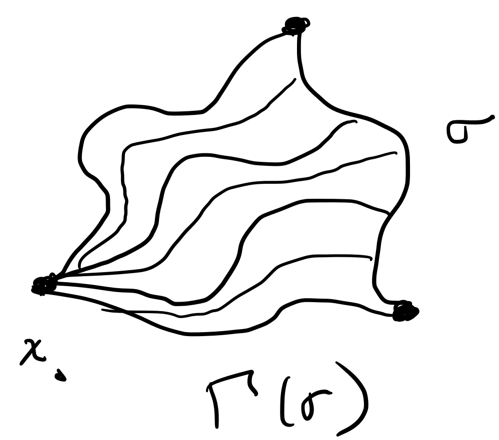
$g : pt \longrightarrow X$ the inclusion of x_0. Since

$$f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_{pt})_* = \mathrm{id}_{H_*(pt)},$$

it suffices to show that $g_* \circ f_* = \mathrm{id}_{H_*(pt)}$.

Given $\sigma : \Delta^n \longrightarrow X$, define $\Gamma(\sigma) : \Delta^{n+1} \longrightarrow X$ by

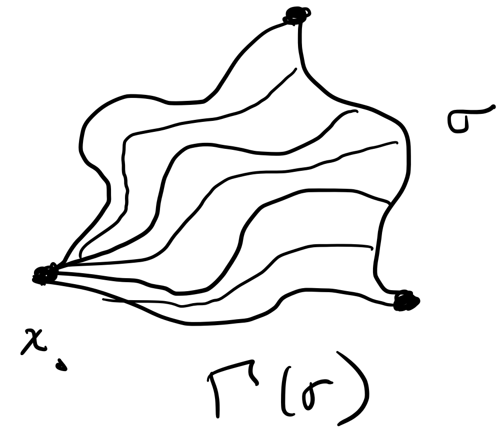
$$\Gamma(\sigma)(t_0, \dots, t_{n+1}) = \begin{cases} H\left(\sigma\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right), t_0\right), & t_0 \neq 1 \\ x_0, & t_0 = 1 \end{cases}$$

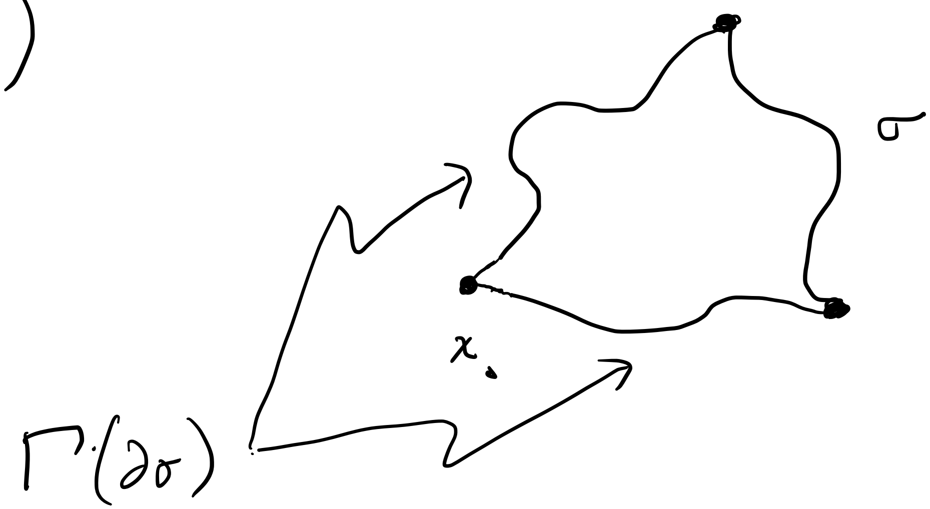


We obtain $\Gamma : C_*(X) \to C_*(X)$ with $\Gamma(C_n(X)) \subseteq C_{n+1}(X)$, and

$$\begin{aligned}\partial\Gamma(\sigma) &= \sum_{i=0}^{n+1}(-1)^i\,\Gamma(\sigma)\circ\eta_i\\ &= \sigma + \sum_{i=1}^{n+1}(-1)^i\,\Gamma(\sigma)\circ\eta_i\\ &= \sigma + \sum_{i=1}^{n+1}(-1)^i\,\Gamma(\sigma\circ\eta_{i-1})\\ &= \begin{cases}\sigma - \Gamma(\partial\sigma), & n>0\\ \sigma - g\circ f\circ\sigma\end{cases}\end{aligned}$$

$$\begin{tikzcd} \Delta^n \arrow[d, "\sigma"] \\ X \arrow[d, "f"] \\ pt \arrow[d, "g"] \\ X \end{tikzcd}$$





$$\Longrightarrow\quad \partial\Gamma + \Gamma\partial = \mathrm{id}_{C_*(X)} - \varepsilon,$$

where $\varepsilon : C_*(X) \to C_*(X)$ is defined by requiring that $\varepsilon|_{C_n(X)} = 0$ for $n>0$ and

$\varepsilon\big|_{C_0(X)} = C_*(g\circ f)$. It follows that ε is a chain map; moreover, given $c \in Z_n$,

$$c - \varepsilon(c) = \partial\Gamma(c) + \Gamma\overset{0}{\cancel{\partial c}} \in B_n,$$

so ε induces the identity on homology. But ε clearly also induces $(g\circ f)_* = g_*\circ f_*$ so $g_*\circ f_* = \mathrm{id}_{H_*(X)}$, as desired.

$\square$