Last time

- Homology
- $H_*(pt)$
- Chain maps, quasi-isomorphisms
- Induced maps $\quad f_* : H_*(X) \to H_*(Y)$
- Contractibility $\quad H_*(X) \xrightarrow{\cong} H_*(pt)$

We extract two definitions for later use.

<u>Def</u> Let $f, g : X \to Y$ be maps. A <u>homotopy</u> from f to g is a map $H : X \times [0,1] \to Y$ such that

$$H(x,t) = \begin{cases} f(x) & \text{if } t=0 \\ g(x) & \text{if } t=1 \end{cases}$$

We say that f + g are <u>homotopic</u>.

<u>Def</u> Let $\varphi, \psi : C \to D$ be chain maps. A homomorphism $\Gamma : C \to D$ is a <u>chain homotopy</u> from φ to ψ if

(1) $\Gamma(C_n) \subseteq D_{n+1}$, and

(2) $\partial\Gamma + \Gamma\partial = \varphi - \psi$.

We say that φ and ψ are <u>chain homotopic</u>.

Prop Chain homotopic chain maps induce the same homomorphism on homology.

Proof Exercise. $\square$

Guess Homotopic maps induce chain homotopic chain maps.

Loose end What about H_0?

Prop There is a canonical natural isomorphism

$$H_0(X) \cong \mathbb{Z}\langle \pi_0(X) \rangle$$

where $\pi_0(X)$ is the set of path components of X.

The word "natural" in the previous statement refers to the commuting diagram

$$\begin{tikzcd} H_0(X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & \mathbb{Z}\langle \pi_0(X) \rangle \arrow[d] & {[x]} \arrow[d, mapsto] \\ H_0(Y) \arrow[r, "\cong"] & \mathbb{Z}\langle \pi_0(Y) \rangle & {[f(x)]} \end{tikzcd}$$

for every map $f : X \to Y$.

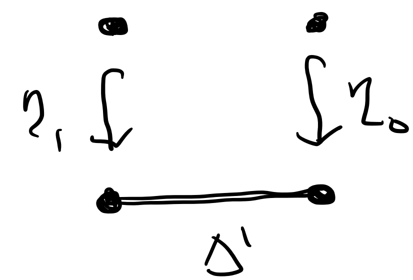
<u>Proof</u> Since $\Delta^0 = \{1\}$, the first homomorphism in the following composite is an isomorphism:

$$\sigma : \Delta^0 \to X \longmapsto \sigma(1)$$

$$x \longmapsto [x]$$

$$\begin{tikzcd}
C_0(X) \arrow[r, "\cong"] \arrow[d, two heads] & \mathbb{Z}\langle X \rangle \arrow[r, two heads] & \mathbb{Z}\langle \pi_0(X) \rangle \\
H_0(X) \arrow[urr, dashed, "\varphi"] & &
\end{tikzcd}$$

To see that the dashed filler φ exists, we note that, for $\sigma : \Delta^1 \to X$,



$$\partial\sigma = \sigma \circ \eta_0 - \sigma \circ \eta_1 \longmapsto \sigma(1) - \sigma(0) \longmapsto 0$$

Since σ is a path from $\sigma(0)$ to $\sigma(1)$.

It is immediate that φ is surjective. For injectivity, suppose $\varphi(c) = 0$, where

above $\quad c = \sum_{i=1}^{m} n_i x_i .$

WLOG the x_i in a common path component, so

$$0 = \varphi(c) = \sum_{i=1}^{m} n_i [x_i] \Rightarrow \sum_{i=1}^{m} n_i = 0.$$

Choose x_0 in this component and paths σ_i from x_0 to x_i for each i. Then

$$\partial\left(\sum_{i=1}^{m} n_i \sigma_i\right) = \sum_{i=1}^{m} n_i x_i - \left(\sum_{i=1}^{m} n_i\right) x_0 = c.$$

Thus, the kernel of the top composite is $B_0(X)$, so φ is injective. $\square$

<u>Exercise</u> Adapt the argument to show that

$$H_n(X) \cong \bigoplus_{C \in \pi_0(X)} H_n(C).$$

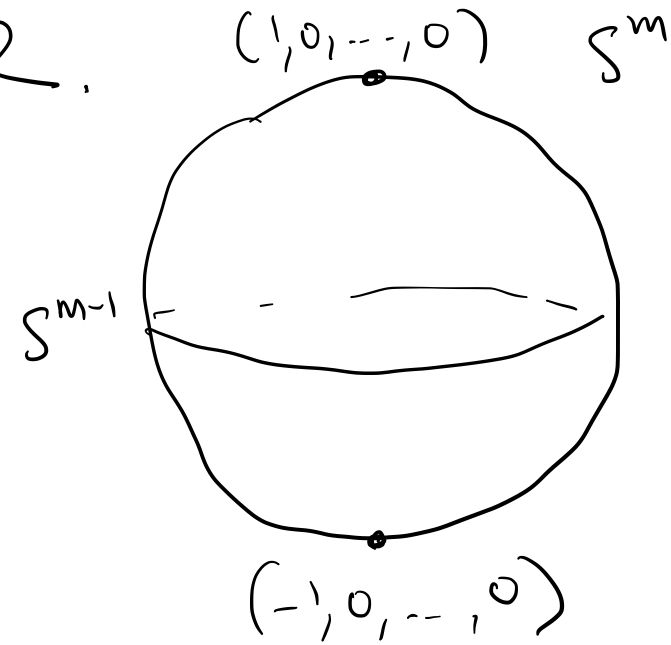
Objection Nontrivial computations seem impossible.

Goal Compute $H_n(S^m)$, $\forall n, m \geq 0$

Observation Via stereographic projection,

$$U_+ := S^m \setminus \{(1,0,\dots,0)\} \cong \mathbb{R}^m \cong S^m \setminus \{(-1,0,\dots,0)\} = U_-,$$

and $U_+ \cap U_- \cong \mathbb{R}^m \setminus \{0\} \cong S^{m-1} \times \mathbb{R}$.



We also write

$$S^m \cong \mathbb{R}^m \coprod_{S^{m-1}\times\mathbb{R}} \mathbb{R}^m .$$

Guess $H_\star(S^m \times \mathbb{R}) \xrightarrow{\cong} H_\star(S^m)$

From before, we know $H_*(\mathbb{R}^n)$ and $H_*(S^0)$, so an inductive argument suggests itself. The missing ingredient is a way of calculating the homology of $U \cup V$ from that of U, V, and $U \cap V$ (gluing).