Last time

- (chain) homotopies
- $H_0(X) \cong \mathbb{Z}\langle \pi_0(X) \rangle$
- **Goal** $H_*(S^m)$

Observation $S^m = U_+ \cup U_-$, where

$$U_\pm = S^m \setminus \{(\pm 1, 0, \ldots, 0)\},$$

and $U_\pm \cong \mathbb{R}^m$, $U_+ \cap U_- \cong S^{m-1} \times \mathbb{R}$.

Guess $H_*(S^m \times \mathbb{R}) \xrightarrow{\cong} H_*(S^m)$

<u>Def</u> Let $i : A \hookrightarrow X$ be the inclusion of a subspace.

(1) A <u>retraction</u> of i is a map $r : X \to A$ such that $r \circ i = id_A$. If such exists, A is called a <u>retract</u> of X.

(2) If in addition $i \circ r$ is homotopic to id_X, we say that A is a <u>deformation retract</u>.

<u>Ex</u> $X = S^{n-1} \times \mathbb{R}$, $A = S^{n-1} \times \{0\}$, $H((x,s),t) = (x, st)$.

<u>Main Dream 1</u> If $A \subseteq X$ is a <u>deformation retract</u>, then $H_*(A) \to H_*(X)$ is an isomorphism.

Consider the general situation where $X = U \cup V$, or

$$X = U \coprod_{U \cap V} V.$$

$$\begin{tikzcd} U \cap V \arrow[r, "j'"] \arrow[d, "i'"'] & U \arrow[d, "i"] \\ V \arrow[r, "j"] & X \end{tikzcd}$$

<u>Observation</u> The kernel of the homomorphism

$$C_*(U) \oplus C_*(V) \xrightarrow{i_\# + j_\#} C_*(X)$$

$\boxed{\begin{matrix} i_\# \\ \shortparallel \\ C_*(i) \\ \text{etc.} \end{matrix}}$

is the image of the homomorphism

$$C_*(U \cap V) \xrightarrow{(i'_\#, -j'_\#)} C_*(U) \oplus C_*(V).$$

<u>Observation</u> The latter homomorphism is injective.

<u>Observation</u> The former homomorphism is not surjective in general. Its image is

$$C_*(\mathcal{U}+\mathcal{V}) := \mathrm{Span}(\mathrm{im}(i_\#), \mathrm{im}(j_\#)).$$

We have a <u>short exact sequence</u>

$$0 \to C_*(\mathcal{U}\cap\mathcal{V}) \to C_*(\mathcal{U}) \oplus C_*(\mathcal{V}) \to C_*(\mathcal{U}+\mathcal{V}) \to 0.$$

<u>Def</u> The pair of composable homomorphisms

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is called <u>exact</u> if $\ker(\psi) = \mathrm{im}(\varphi)$. A longer sequence of composable homomorphisms is exact if each consecutive pair is exact.

Exercise $0 \to A \xrightarrow{\varphi} B$ is exact iff φ is injective.

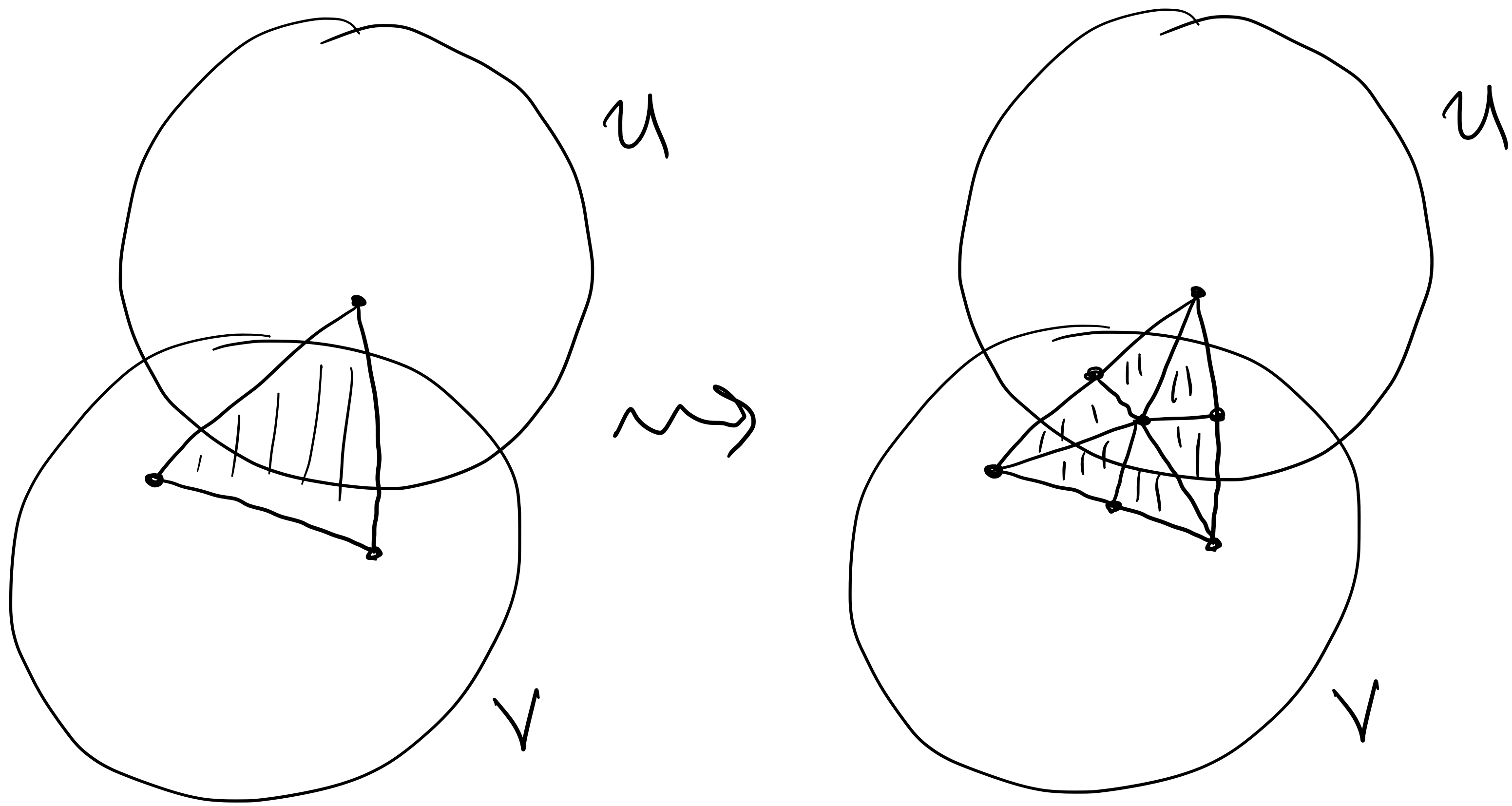
Exercise $B \xrightarrow{\psi} C \to 0$ is exact iff ψ is surjective.

Def A **short exact sequence** is an exact sequence of the form

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0.$$

Main Dream 2 The inclusion $C_*(\mathcal{U}+\mathcal{V}) \longrightarrow C_*(X)$ is a quasi-isomorphism.

Rationale Subdivision.



$\sigma \notin C_*(U+V)$ $\qquad$ $\sum_i \pm \sigma_i \in C_*(U+V)$

Assuming this dream is true, the question becomes one of understanding the homology of short exact sequences of chain complexes.

<u>General situation</u> Suppose we have the exact sequence

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

where A, B, and C are chain complexes and φ and ψ are chain maps.

<u>Observation</u> $\psi_*: H_n(B) \to H_n(C)$ is not necessarily surjective. Indeed, an element in the target lies in the image of ψ_* if and only if it is represented by a cycle with a preimage that is <u>also</u> a cycle.

In other words, given $[c] \in H_n(C)$, the obstruction to $[c]$ lying in im ψ_* is $\partial\tilde{c}$, where $\tilde{c} \in \psi^{-1}(c)$

$$\begin{array}{ccccccccc}
 & & \varphi^{-1}(\partial\tilde{c}) & \longmapsto & \partial\tilde{c} & \longmapsto & 0 & & \\
 & & & & \tilde{c} & \longmapsto & c & & \\
0 & \to & A & \to & B & \to & C & \to & 0
\end{array}$$

<u>Observation</u> $\partial\tilde{c} \in \mathrm{im}(\varphi)$, and $\varphi^{-1}(\partial\tilde{c})$ is a cycle.

<u>Thm</u> The assignment $[c] \longmapsto [\varphi^{-1}(\partial\tilde{c})]$ is a well-defined homomorphism

$$\delta : H_n(C) \longrightarrow H_{n-1}(A),$$

and the sequence

$$\cdots \xrightarrow{\delta} H_n(A) \xrightarrow{\varphi_*} H_n(B) \xrightarrow{\psi_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{\varphi_*} \cdots$$

is exact.

Proof First, $\tilde{c}$ exists by surjectivity of ψ, and $\psi(\partial\tilde{c}) = \partial\psi(\tilde{c}) = \partial c = 0$ by assumption, so $\varphi^{-1}(\partial\tilde{c}) \neq \emptyset$. By injectivity of φ, $\varphi^{-1}(\partial\tilde{c})$ is a single element, which is a cycle since φ is injective and

$$\varphi(\partial\varphi^{-1}(\partial\tilde{c})) = \partial\varphi\varphi^{-1}(\partial\tilde{c}) = \partial^2\tilde{c} = 0.$$

Thus, our formula parses, so it remains to show that $\delta[c]$ is independent of the choice of representative c and preimage $\tilde{c}$. For the second, supposing that $\psi(\bar{c}) = c$, we have $\tilde{c} - \bar{c} = \varphi(a)$ for $a \in A$ by exactness, so

$$\varphi^{-1}(\partial\tilde{c}) - \varphi^{-1}(\partial\bar{c}) = \varphi^{-1}(\partial(\tilde{c}-\bar{c}))$$
$$= \varphi^{-1}(\partial\varphi(a))$$
$$= \varphi^{-1}\varphi(\partial a)$$
$$= \partial a$$

$$\Rightarrow [\varphi^{-1}(\partial\tilde{c})] = [\varphi^{-1}(\partial\bar{c})].$$

For the first, it suffices to show that $\varphi^{-1}(\partial\tilde{c}) = 0$ if $c = \partial c'$. By surjectivity of ψ, $c' = \psi(b)$ and

$$\psi(\partial b) = \partial\psi(b) = \partial c' = c,$$

so we may take $\tilde{c} = \partial b$ by the previous argument, whence

$$\varphi^{-1}(\partial\tilde{c}) = \varphi^{-1}(\partial^2 b) = 0.$$

For exactness, we note first that $\psi_* \circ \varphi_* = 0$, since $\psi \circ \varphi = 0$; that $\varphi_* \circ \delta = 0$, since $\varphi(\varphi^{-1}(\partial\tilde{c}))$ is a boundary; and that $\delta \circ \psi_* = 0$, since $\varphi^{-1}(\partial\tilde{c}) = 0$ if $\tilde{c}$ is a cycle.

There are now three points to verify.

For the containment $\ker(\delta) \subseteq \operatorname{im}\psi_*$, suppose that $\delta([c]) = 0$, so $\varphi^{-1}(\partial\tilde{c}) = \partial a$.

Then $\psi(\tilde{c} - \varphi(a)) = c - \psi(\varphi(a)) = c$, and $\partial(\tilde{c} - \varphi(a)) = \partial\tilde{c} - \varphi(\partial a) = 0$ by injectivity of φ, so $c = \psi_*([\tilde{c} - \varphi(a)])$. For the containment $\ker(\varphi_*) \subseteq \operatorname{im}(\delta)$, suppose $\varphi_*([a]) = 0$, so $\varphi(a) = \partial b$. We claim that $\psi(b)$ is a cycle and $\delta([\psi(b)]) = [a]$. Indeed, we have $\partial\psi(b) = \psi(\partial b) = \psi(\varphi(a)) = 0$, and $\varphi^{-1}(\partial\widetilde{\psi(b)}) = \varphi^{-1}(\partial b) = a$. Finally, for the containment $\ker(\psi_*) \subseteq \operatorname{im}(\varphi_*)$, suppose that $\psi(b) = \partial c$. By surjectivity, we may write $c = \psi(b')$, and

$$\psi(b - \partial b') = \partial c - \partial\psi(b') = 0,$$

so $b - \partial b' = \varphi(a)$ by exactness. Since

$[b]=[b-\partial b']$, it suffices to show that a is a cycle, which follows from injectivity of φ and the calculation

$$\varphi(\partial a)=\partial\varphi(a)=\partial b-\partial\partial b'=0.$$

$\square$