Last time

- Dream 1: deformation retracts
- Dream 2: subdivision
- Exact sequences

Applying the theorem to the exact sequence

$$0 \to C_*(U\cap V) \to C_*(U)\oplus C_*(V) \to C_*(U+V) \to 0 \ :$$

Corollary (Mayer-Vietoris) If Main Dream 2 holds for $X = U\cup V$, there is a long exact sequence

$$\cdots \to H_n(U\cap V) \xrightarrow{(j'_*,-i'_*)} H_n(U)\oplus H_n(V) \xrightarrow{i_*+j_*} H_n(X) \xrightarrow{\delta} H_{n-1}(U\cap V) \to \cdots .$$

Rmk We will prove this dream when $X \subseteq \mathring{U} \cup \mathring{V}$.

Now we can accomplish our computational goal. It is convenient to introduce a slight variant of homology.

Def The reduced homology of X is

$$\tilde{H}_*(X) = \ker(H_*(X) \to H_*(pt)) \subseteq H_*(X)$$
$$= H_*(\tilde{C}_*(X)),$$

where $\tilde{C}_*(X) := \ker(C_*(X) \to C_*(pt))$.

Rmk For $n>0$, $\tilde{H}_n(X) = H_n(X)$, and $H_0(X) \cong \tilde{H}(X) \oplus \mathbb{Z}$ after choosing a basepoint (exercise).

Rmk The Mayer-Vietoris sequence is still exact after restricting to reduced homology (exercise).

Corollary Assuming Main Dreams 1 and 2,

$$\tilde{H}_n(S^m) \cong \begin{cases} \mathbb{Z} & n=m \\ 0 & \text{otherwise.} \end{cases}$$

Proof We proceed by induction on m. For $m=0$, we have the commuting diagram

$$\begin{tikzcd}
H_0(S^0) \arrow[r, "\cong"] \arrow[d] & \mathbb{Z}\langle \pi_0(S^0) \rangle \arrow[d] \arrow[r, phantom, "\cong"] & \mathbb{Z} \oplus \mathbb{Z} & (1,0) \arrow[d, mapsto] & (0,1) \arrow[d, mapsto] \\
H_0(pt) \arrow[r, "\cong"] & \mathbb{Z}\langle \pi_0(pt) \rangle \arrow[r, phantom, "\cong"] & \mathbb{Z} & 1 & 1
\end{tikzcd}$$

and the kernel of the righthand map is the subgroup generated by $(1,-1)$, which is isomorphic to $\mathbb{Z}$, so $\tilde{H}_0(S^0) \cong \mathbb{Z}$. For $n>0$, $\tilde{H}_n(S^0) = H_n(S^0) \cong H_n(pt) \oplus H_n(pt) = 0$, so the claim holds in this case.

For the induction step, we have the exact sequence (using Main Dream 2)

$$\cdots \to \cancelto{0}{\tilde{H}_n(\mathbb{R}^m)}^{\oplus 2} \longrightarrow \tilde{H}_n(S^m) \xrightarrow{\delta} \tilde{H}_{n-1}(S^{m-1} \times \mathbb{R}) \to \cancelto{0}{\tilde{H}_{n-1}(\mathbb{R}^m)}^{\oplus 2} \to \cdots,$$

so $\tilde{H}_n(S^m) \cong \tilde{H}_{n-1}(S^{m-1} \times \mathbb{R}) \cong \tilde{H}_{n-1}(S^{m-1})$ by Main Dream 1. $\square$

One could imagine approaching this problem from a different perspective.

<u>Observation</u> $S^m \cong D^m/\partial D^m$, D^m is contractible, and $\partial D^m \cong S^{m-1}$!

Another inductive argument suggests itself, premised on understanding the homology of quotients by subspaces rather than gluing constructions.

Consider the general situation of a subspace $A \subseteq X$. The quotient X/A has a canonical basepoint given by the image of A, and the following diagram commutes:

$$\begin{tikzcd}
C_*(A) \arrow[r] \arrow[d] & C_*(X) \arrow[r] \arrow[d] & C_*(X)/C_*(A) =: C_*(X,A) \arrow[d, dashed] \\
C_*(pt) \arrow[r] & C_*(X/A) \arrow[r] & C_*(X/A)/C_*(pt) \cong \tilde{C}_*(X/A).
\end{tikzcd}$$

<u>Main Dream 3</u> The map $C_*(X,A) \to \tilde{C}_*(X/A)$ is a quasi-isomorphism.

In this case, the homology of the quotient coincides with <u>relative</u> homology.

Def The *relative homology* of the pair (X, A), denoted $H_*(X, A)$, is the homology of the chain complex $C_*(X, A)$.

The purpose of relative homology is that it fits into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \cdots.$$

This sequence gives another route to the same calculations.

Corollary Assuming Main Dream 3,

$$\tilde{H}_n(S^m) \cong \begin{cases} \mathbb{Z}, & n=m \\ 0 & \text{otherwise.} \end{cases}$$

Proof The long exact sequence in relative homology for the pair $(D^m, \partial D^m)$ is

$$\cdots \to \cancelto{0}{H_n(D^m)} \to H_n(D^m, \partial D^m) \to H_{n-1}(\partial D^m) \to \cancelto{0}{H_{n-1}(D^m)} \to \cdots,$$

so $H_{n-1}(S^{m-1}) = H_{n-1}(\partial D^m) \cong H_n(D^m, \partial D^m) \cong H_n(S^m)$ by Main Dream 3 for $n>1$. $\square$

Rmk We will prove Main Dream 3 when A is a deformation retract of an open in X.