Last time

- Mayer-Vietoris
- $H_*(S^n)$
- Reduced/relative

Dreams/Debt

(1) Deformation retracts
$(H_*(A) \xrightarrow{\cong} H_*(X))$

(2) Subdivision
$(C_*(U+V) \xrightarrow{\sim} C_*(U \cup V))$

(3) Quotients
$(H_*(X,A) \xrightarrow{\sim} \tilde{H}_*(X/A))$

Before turning to applications, our main interest, we clarify the nature of our debt.

<u>Homotopy thm</u> $f \simeq g \Longrightarrow C_*(f) \simeq C_*(g)$.

<u>Cor</u> $f \simeq g \Longrightarrow f_* = g_*$.

Def A map $f: X \to Y$ is a homotopy equivalence if there is a map $g: Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$ ("homotopy inverse")

Exercise Assuming the Homotopy Theorem, a homotopy equivalence induces an isomorphism on homology.

A similar result follows for relative homology.

Def A map of pairs $f: (X, A) \to (Y, B)$ is a map $f: X \to Y$ such that $f(A) \subseteq B$.

Exercise Maps of pairs induce homomorphisms on relative homology.

Prop If $f:(X,A) \to (Y,B)$ is a map of pairs such that $f: X \to Y$ and $f|_A : A \to B$ are homotopy equivalences, then f induces an isomorphism on homology.

Lemma (Five lemma) Suppose both rows are exact in the commutative diagram of Abelian groups

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}.$$

(1) If β + δ are surjective and ε is injective, then γ is surjective.

(2) If β & δ are injective and α is surjective, then γ is injective.

In particular, if $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, then so is γ.

Exercise Prove the five lemma.

Exercise Use the five lemma to prove the proposition.

We turn to subdivision. Given a collection $\theta = \{U_\alpha\}_{\alpha \in A}$ of subspaces of X, write

$$C_n^\theta(X) = \left\{ \sum_{i=1}^m n_i \sigma_i \in C_n(X) \;\middle|\; \forall i, \exists \alpha_i : \operatorname{im}(\sigma_i) \subseteq U_{\alpha_i} \right\}.$$

Exercise $C_*^\theta(X)$ is a subcomplex of $C_*(X)$.

Example $C_*(U+V) = C_*^{\{U,V\}}(X)$.

Subdivision thm If $\{\mathring{U}_\alpha\}_{\alpha\in A}$ is an open cover of X, then $C^{\mathcal{O}}_*(X) \subseteq C_*(X)$ is a chain homotopy equivalence.

This theorem obviously covers our second debt. Less obviously, it also covers our third.

Cor (Excision) If $X = \mathring{U} \cup \mathring{V}$, then

$$H_*(V, U\cap V) \xrightarrow{\cong} H_*(X, U).$$

Rmk A common equivalent formulation is

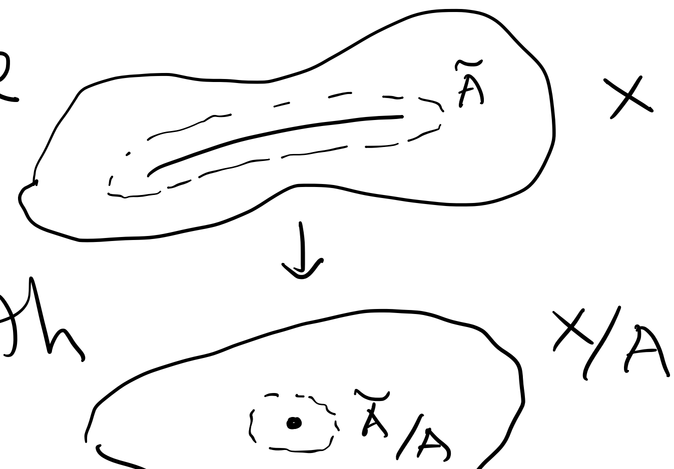
$H_*(X\setminus Z, A\setminus Z) \xrightarrow{\cong} H_*(X,A)$ for $\overline{Z} \subseteq \mathring{A}$.

Cor If $A \subseteq X$ is a deformation retract of an open set $A \subseteq \tilde{A} \subseteq X$, then $H_*(X,A) \xrightarrow{\cong} \tilde{H}_*(X/A)$.

Proof Applying relative homology to the commutative diagram of pairs

$$\begin{tikzcd}
(X,A) \arrow[r, hook, "1"] \arrow[d] & (X,\tilde{A}) \arrow[d] & (X \setminus A, \tilde{A} \setminus A) \arrow[l, "3"'] \arrow[d, "5"] \\
(X/A, A/A) \arrow[r, hook, "2"] & (X/A, \tilde{A}/A) & (X/A \setminus A/A, \tilde{A}/A \setminus A/A) \arrow[l, hook', "4"']
\end{tikzcd}$$

the first and second rows induce isomorphisms by homotopy invariance, the third and fourth by excision, and the fifth is



a homeomorphism of pairs.

$\square$

<u>Proof of excision assuming subdivision</u> Consider the commuting diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C_*(U\cap V) \arrow[r] \arrow[d] & C_*(V) \arrow[r] \arrow[d] & C_*(V)/C_*(U\cap V) \arrow[r] \arrow[d, "③"] & 0 \\
0 \arrow[r] & C_*(U) \arrow[r] \arrow[d, equal] & C_*(U+V) \arrow[r] \arrow[d, "①"] & C_*(U+V)/C_*(U) \arrow[r] \arrow[d, "②"] & 0 \\
0 \arrow[r] & C_*(U) \arrow[r] & C_*(X) \arrow[r] & C_*(X)/C_*(U) \arrow[r] & 0
\end{tikzcd}$$

The first map is a quasi-isomorphism by subdivision, so the second is so as well by the five lemma. Since the third is an isomorphism on the nose, the claim follows. $\square$