Last time

- Debt clarification
- Five lemma
- Subdivision $\Rightarrow$ excision $\Rightarrow$ quotients

Homotopy thm If f and g are homotopic, then $f_\#$ and $g_\#$ are chain homotopic.

Given a homotopy $H: X \times [0,1] \to Y$ from f to g, we seek to produce a homomorphism $P: C_*(X) \to C_*(Y)$ such that

(1) $P(C_n(X)) \subseteq C_{n+1}(Y)$, and

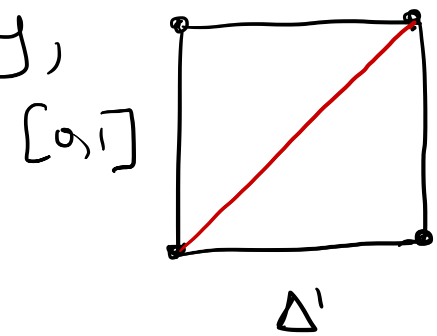
(2) $\partial P + P \partial = g_\# - f_\#$.

<u>Idea</u> Given a singular simplex $\sigma: \Delta^n \to X$, the homotopy H gives a map

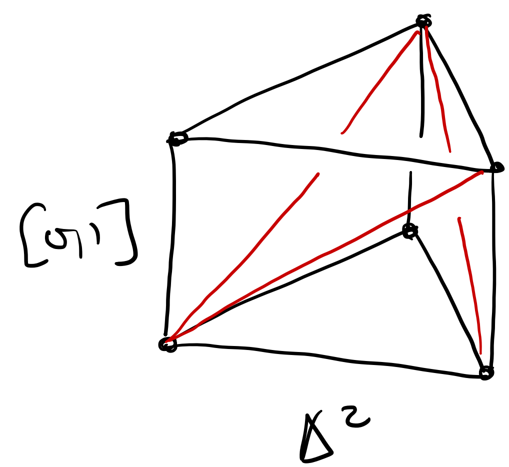
$$\Delta^n \times \Delta^1 = \Delta^n \times [0,1] \xrightarrow{\sigma \times id} X \times [0,1] \xrightarrow{H} Y,$$

which is <u>almost</u> a singular $(n+1)$-simplex.

More specifically, after subdividing, it is a union of singular simplices.



To make this idea precise, it is useful to make a change of coordinates in the standard simplex.



<u>Lemma/exercise</u> The function

$$\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$$

$$(t_0, \dots, t_n) \longmapsto (s_1, \dots, s_n)$$

given by $s_i(t_0, \dots, t_n) = \sum_{j=0}^{i-1} t_j$ restricts to a homeomorphism

$$\Delta^n \cong \{(s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 \le s_1 \le \dots \le s_n \le 1\}$$

<u>Ex</u> For $n=1$, this is the standard linear homeomorphism $\Delta^1 \cong [0,1]$.

We abuse notation by writing Δ^n for this subspace of $\mathbb{R}^n$ as well.

Construction For $1 \le i \le n+1$, define $P_i : \Delta^{n+1} \to \Delta^n \times [0,1]$ by

$$P_i(s_1, \dots, s_{n+1}) = ((s_1, \dots, \hat{s}_i, \dots, s_{n+1}), s_i).$$

Given a homotopy $H : X \times [0,1] \to Y$, define $P : C_n(X) \to C_{n+1}(Y)$ by

$$P\sigma = \sum_{i=1}^{n+1} (-1)^i H \circ (\sigma \times id) \circ P_i .$$

Lemma Writing $\Delta^n = \{(s_1, \dots, s_n) \in \mathbb{R}^n \mid 0 \le s_1 \le \dots \le s_n \le 1\}$, $\eta_i : \Delta^n \to \Delta^{n+1}$ is given by the formula

$$\eta_i(s_1,\dots,s_n) = \begin{cases} (0,s_1,\dots,s_n), & i=0 \\ (s_1,\dots,s_i,s_i,\dots,s_n), & 0<i<n+1 \\ (s_1,\dots,s_n,1), & i=n+1 \end{cases}.$$

Proof Exercise. $\square$

Proof of thm From the lemma, we have

$$p_i \circ \eta_j = \begin{cases} \mathrm{id}\times\{0\}, & j=0,\ i=1 \\ \mathrm{id}\times\{1\}, & i=j=n+1 \\ (\eta_{j-1}\times \mathrm{id})\circ p_i & j>i \\ (\eta_j \times \mathrm{id})\circ p_{i-1} & i>j+1 \\ p_{i+1}\circ \eta_j & i=j<n+1 \end{cases},$$

$$\begin{tikzcd} \Delta^n \arrow[d, hook] \\ \Delta^{n+1} \arrow[d] \\ \Delta^n\times[0,1] \end{tikzcd}$$

so

$$\partial P_\sigma = \partial\left[\sum_{i=1}^{n+1} (-1)^i H\circ(\sigma\times id)\circ p_i\right]$$

$$= \sum_{j=0}^{n+1}\sum_{i=1}^{n+1} (-1)^{i+j} H\circ(\sigma\times id)\circ p_i\circ\eta_j$$

$$= \sum_{j>i} + \sum_{i>j+1} + \sum_{i=j} + \sum_{i=j+1}$$

$$=: A + B + C + D.$$

We will show that $A+B = -P_{\partial\sigma}$ and $C+D = g_\# - f_\#$. For the first claim we have by the above calculation

$$A+B = \sum_{1\le i<j\le n+1} (-1)^{i+j} H\circ(\sigma\times id)\circ(\eta_{j-1}\times id)\circ P_i$$

$$+ \sum_{1\le j+1<i\le n+1} (-1)^{i+j} H\circ(\sigma\times id)\circ(\eta_j\times id)\circ P_{i-1}$$

$$= \sum_{1\le i\le j\le n} (-1)^{i+j+1} H\circ(\sigma\circ\eta_j\times id)\circ P_i$$

$$+ \sum_{1\le j+1\le i\le n} (-1)^{i+j+1} H\circ(\sigma\circ\eta_j\times id)\circ P_i$$

$$= -P\left[\sum_{j=0}^{n} (-1)^j \sigma\circ\eta_j\right]$$

$$= -P\partial\sigma\,.$$

For the second,

$$C+D = \sum_{i=1}^{n+1} (-1)^{2i} H\circ(\sigma\times\mathrm{id})\circ P_i \circ \eta_i + \sum_{j=0}^{n} (-1)^{2j+1} H\circ(\sigma\times\mathrm{id})\circ P_{j+1}\circ\eta_j$$

$$= \sum_{i=1}^{n} (-1)^{2i} H\circ(\sigma\times\mathrm{id})\circ P_i\circ\eta_i - \sum_{j=1}^{n} (-1)^{2j} H\circ(\sigma\times\mathrm{id})\circ P_j\circ\eta_j$$

$$+ H\circ(\sigma\times\mathrm{id})\circ(\mathrm{id}\times\{1\}) - H\circ(\sigma\times\mathrm{id})\circ(\mathrm{id}\times\{0\})$$

$$= H\circ(\sigma\times\{1\}) - H\circ(\sigma\times\{0\})$$

$$= g\circ\sigma - f\circ\sigma$$

$$= g_\#\sigma - f_\#\sigma.$$

$\square$