Last time

- Homotopy theorem

Before turning to subdivision, a palate cleanser. With the computation of the homology of spheres, we now have access to many applications. The first few, although simple, illustrate one of the central uses of (co)homology: obstructing the existence of maps with specified properties.

For these purposes, it is convenient to make the following definition.

Thm For every $m > 0$, there is a function

$$\left\{ \begin{matrix} \text{maps} \\ S^m \to S^m \end{matrix} \right\} \longrightarrow \mathbb{Z},$$
$$f \longmapsto \deg(f)$$

called the **degree**, with the following properties.

(1) $\deg(\mathrm{id}_{S^m}) = 1$

(2) $\deg(f \circ g) = \deg(f)\deg(g)$

(3) $f \simeq g \Rightarrow \deg(f) = \deg(g)$

(4) $\deg(f) = 0$ if f is not surjective

(5) The degree of a reflection is -1

(6) $\deg(f) = (-1)^{m+1}$ if f has no fixed points

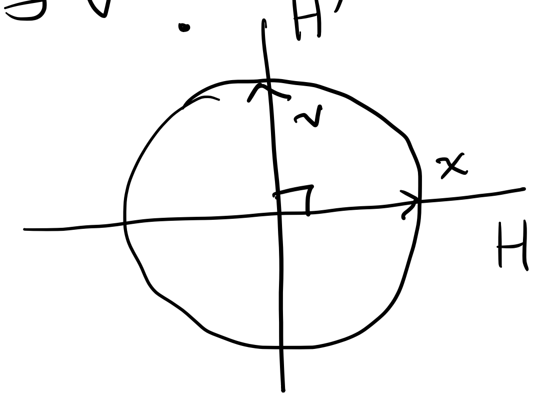
Proof of thm Choose a generator for the free Abelian group $\tilde{H}_m(S^m)$ and define

$\deg(f)$ by $f_*(\alpha) = \deg(f)\cdot\alpha$. Properties (1) and (2) follow from functoriality and (3) from homotopy invariance. For (4), if $x \notin \mathrm{im}(f)$, then f factors as

$$S^m \longrightarrow S^m \setminus \{x\} \subseteq S^m$$

$$S^m \setminus \{x\} \cong \mathbb{R}^m,$$

So f_* factors through $\tilde{H}_*(\mathbb{R}^m) = 0$. For (5), let f denote reflection across the hyperplane H, and choose $v \in H^\perp$ and a hyperplane $H' \ni v$.



We have $(H')^\perp \cap S^m = \{\pm x\}$, $S^m = S^m \setminus \{x\} \cup S^m \setminus \{-x\}$, and

$S^m \setminus \{x\} \cap S^m \setminus \{-x\} \simeq H' \cap S^m \cong S^{m-1}$. Since $f|_{H' \cap S^m}$ is reflection across the hyperplane $H \cap H' \subseteq H'$, which has degree -1 by induction, the conclusion follows from the Mayer-Vietoris sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_m(S^m) & \longrightarrow & H_{m-1}(H' \cap S^m) & \longrightarrow & 0 \\ & & \downarrow f_* & & \downarrow (f|_{H' \cap S^m})_* & & \\ 0 & \longrightarrow & H_m(S^m) & \longrightarrow & H_{m-1}(H' \cap S^m) & \longrightarrow & 0. \end{array}$$

The base case, which is the claim that the map $S^0 \to S^0$ switching the two points induces multiplication by -1 on $\tilde{H}_0(S^0)$, follows from our calculation of that group.

A particular case of (6) now follows: the antipodal map, which is a composite of $m+1$ reflections, has degree $(-1)^{m+1}$ by (2) and (5). By (3), it suffices to note that a map f with no fixed point is homotopic to the antipodal map via the homotopy

$$H(x,t) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|},$$

which is well defined since $(1-t)f(x) - tx = 0$ iff $f(x) = x$ for $x \in S^m$. $\square$

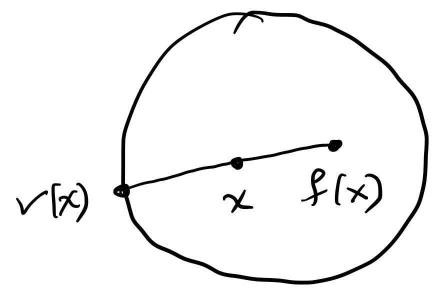
Now, applications.

Cor For $m \geq 0$, the inclusion $i : S^m \subseteq D^{m+1}$ does not admit a retraction.

Proof Since $\tilde{H}_m(D^{m+1}) = 0$, $i_* = 0$, so $\deg(r \circ i) = 0$ for any $r : D^{m+1} \to S^m$. If r is a retraction, $\deg(r \circ i) = \deg(id) = 1$, which is impossible. $\square$

Cor (Brouwer fixed point theorem) Any map $f: D^{m+1} \to D^{m+1}$ has a fixed point.

Picture proof



$\square$

Cor S^m admits a continuous nonvanishing tangent vector field iff m is odd.

Proof Assuming v is such a vector field, we may assume $\|v(x)\| = 1$ for each $x \in S^m$. Since $x \perp v(x)$, $\|\cos(t)x + \sin(t)v(x)\| = 1$, so we

$S^m \xrightarrow{v} \mathbb{R}^{m+1}$

$v(x) \perp x$

obtains a homotopy from id_{S^m} to the antipodal map, whose respective degrees of 1 and $(-1)^{m+1}$ must coincide.

Conversely, if $m = 2k-1$, then $(-x_2, x_1, \dots, -x_{2k-1}, x_{2k})$ is a nonvanishing tangent vector field. $\square$

Cor If m is even and $G \neq \{1\}$ is a group acting continuously and freely on S^m, then $G \cong \mathbb{Z}/2\mathbb{Z}$.

Proof Since $\deg(f_1 \circ f_2) = \deg(f_1)\deg(f_2)$ and $\deg(f) = \pm 1$ if f is a homeomorphism, the assignment

$$\varphi : G \longrightarrow \{\pm 1\}$$

$$g \longmapsto \deg(x \mapsto gx)$$

is a homomorphism. If G acts freely, then $x \mapsto gx$ has no fixed point, so $\varphi(g) = (-1)^{m+1}$ for every $g \neq 1$. If m is even, then, $\ker(\varphi) = \{1\}$, so φ is injective. Since $G \neq \{1\}$, φ is also surjective.

$\square$

We turn now to another type of application, also a hallmark of algebraic topology: hard but simple geometric questions.

One of the oldest examples is the Jordan curve theorem, which states that any embedded circle ("simple closed curve") seperates $\mathbb{R}^2$ (equivalently, S^2) into two regions. We will prove a result generalizing this plausible but difficult fact.

<u>Thm</u> (Generalized Jordan curve theorem)
If $f: S^r \to S^m$ is an embedding, then

$$\tilde{H}_n(S^m, \operatorname{im}(f)) \cong \begin{cases} \mathbb{Z}, & n = m-r-1 \\ 0, & \text{otherwise.} \end{cases}$$