Last time

- Image, kernel, and rank
- Subspaces
- Spans

We saw that the image of a matrix is spanned by its columns, but this description can be very redundant.

Def Consider the list of vectors $\vec{v}_1, \ldots, \vec{v}_r$ in $\mathbb{R}^n$.

(1) The vector $\vec{v}_i$ is redundant if it is

a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_{i-1}$.

(2) The vectors $\vec{v}_1, \ldots, \vec{v}_r$ are linearly independent if none are redundant.

(3) A linear relation among $\vec{v}_1, \ldots, \vec{v}_r$ is an equation of the form

$$c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0}.$$

It is trivial if $c_1 = c_2 = \cdots = c_r = 0$ and nontrivial otherwise.

Having no redundant vectors is the same as having no nontrivial linear relations

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columns of $A$ linearly independent $\Leftrightarrow$ $A\vec{x}=\vec{0}$ has a unique solution $\Leftrightarrow$ $\ker A = \{\vec{0}\}$ $\Leftrightarrow$ $rk(A)=n$

So a collection of more than $m$ vectors in $\mathbb{R}^m$ can't be linearly independent!

columns of $A$ span $\mathbb{R}^m$ $\Leftrightarrow$ $A\vec{x}=\vec{b}$ has a solution for every $\vec{b}$ $\Leftrightarrow$ $\operatorname{im} A = \mathbb{R}^m$ $\Leftrightarrow$ $rk(A)=m$

So a collection of fewer than $m$ vectors in $\mathbb{R}^m$ can't span!

**Def** A **basis** for a subspace $V$ of $\mathbb{R}^n$ is a linearly independent collection of vectors in $V$ that span $V$.

$$\boxed{\text{Every basis for a subspace has the same number of vectors, called the } \underline{\underline{\text{dimension}}}\ \dim V \text{ of the subspace.}}$$

**Ex** $\dim \mathbb{R}^m = m$

Why care about having a basis?

If $\{\vec{v}_1, \dots, \vec{v}_r\}$ is a basis for the subspace $V$, then any vector $\vec{v}$ in $V$ can be written uniquely as a linear combination

$$\vec{v} = c_1\vec{v}_1 + \dots + c_r\vec{v}_r$$