Last time

- Finding bases for kernel and image

We saw that $\dim \ker(A)$ is the number of free variables and $\dim \operatorname{im}(A)$ the number of pivots.

Rank-nullity theorem If A is $m \times n$,

$$\boxed{\dim \operatorname{im}(A) + \dim \ker(A) = n}$$

Consider the example:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$V = \mathrm{Span}(\vec{v}_1, \vec{v}_2)$$

Since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent (and span V by definition) the collection $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for V. Now, if $\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$, then $\vec{x}$ is in V if and only if

Is a linear combination

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

To check, we solve:

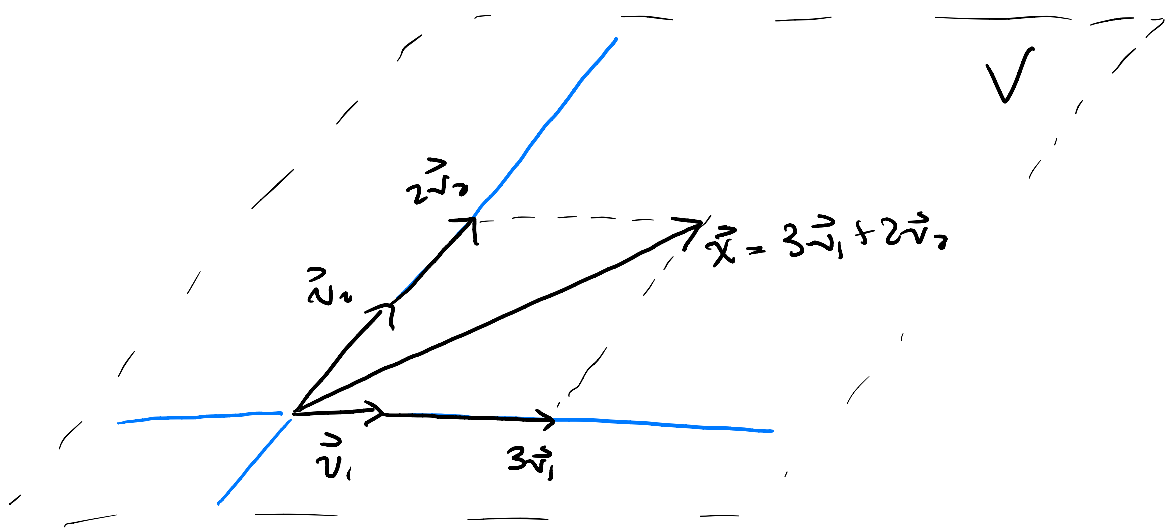
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right],$$

so $c_1 = 3$ and $c_2 = 2$, i.e.,

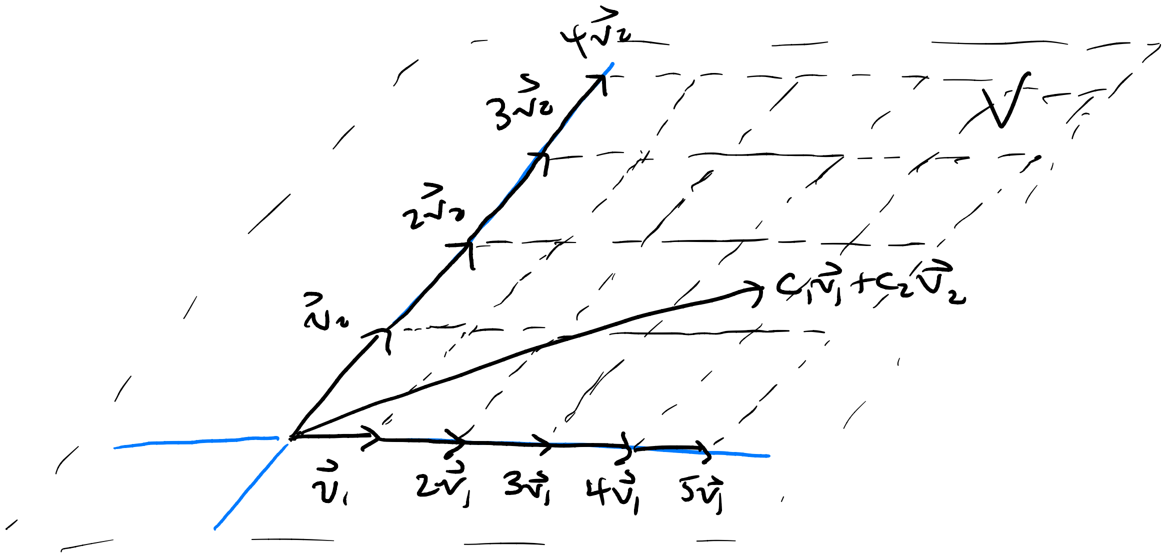
$$\vec{x} = 3\vec{v}_1 + 2\vec{v}_2,$$

and this expression is <u>unique</u>.

Now, since $\mathcal{B}$ has two elements, $\dim V = 2$, i.e., V is a plane in $\mathbb{R}^3$. Schematically,



We can see that the lines parallel to $\vec{v}_1$ and $\vec{v}_2$ function as coordinate axes for V, even though they aren't perpendicular!



Any vector in V is uniquely described by the numbers c_1 and c_2.

Def Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_r\}$ be a basis for a subspace V of $\mathbb{R}^m$. Given a vector $\vec{x}$ in V, the coordinates of $\vec{x}$ in the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\vec{x}$) are the scalars $c_1, \ldots, c_r$ such that

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_r\vec{v}_r.$$

We write

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}.$$

Ex In the earlier example, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Ex If $\mathcal{B}$ is the standard basis for $\mathbb{R}^m$, then

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

The relationship between the usual coordinates and $\mathcal{B}$-coordinates is given by the equation

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_r\vec{v}_r$$

$$= \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_r \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}$$

$$= S[\vec{x}]_{\mathcal{B}},$$

where $S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_r \end{bmatrix}$. If $V = \mathbb{R}^m$, then S is square of rank m, hence invertible.

If $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ is a basis for $\mathbb{R}^m$, then

$$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x},$$

where $S = [\vec{v}_1 \cdots \vec{v}_m]$.

<u>Ex</u> Let L be the line in $\mathbb{R}^2$ parallel to $\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (chosen so $|\vec{u}_1| = 1$). The perpendicular line is parallel to $\vec{u}_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$, so $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ is a basis

for $\mathbb{R}^2$. We have

$$S = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$\Longrightarrow S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix},$$

so if $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\begin{aligned} [\vec{v}]_{\mathcal{B}} &= S^{-1}\vec{v} \\ &= \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 3x+4y \\ -4x+3y \end{bmatrix} \\ &= \begin{bmatrix} \vec{v}\cdot u_1 \\ \vec{v}\cdot u_2 \end{bmatrix}. \end{aligned}$$