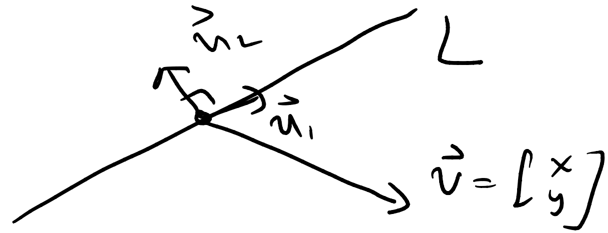
Last time

- Coordinates

The coordinates of the vector $\vec{x}$ in the basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_r\}$ are the scalars $c_1, \dots, c_r$ such that $\vec{x} = c_1\vec{v}_1 + \dots + c_r\vec{v}_r$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}.$$

Ex $\quad \vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$



$\vec{u}_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ a basis for $\mathbb{R}^2$

$$S = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \qquad S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3x+4y \\ -4x+3y \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1 \cdot \vec{v} \\ \vec{u}_2 \cdot \vec{v} \end{bmatrix}.$$

On the other hand, $\mathrm{proj}_L(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1$,

so
$$[\mathrm{proj}_L(\vec{v})]_{\mathcal{B}} = [(\vec{u}_1 \cdot \vec{v})\vec{u}_1]_{\mathcal{B}}$$
$$= (\vec{u}_1 \cdot \vec{v})[\vec{u}_1]_{\mathcal{B}}$$
$$= (\vec{u}_1 \cdot \vec{v})\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \vec{u}_1 \cdot \vec{v} \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}[\vec{v}]_{\mathcal{B}}$$

<u>Upshot</u> Projection is simplified by working in coordinates adapted to it.

$$
\begin{tikzcd}
\vec{v} \arrow[r, mapsto, "A = \mathrm{proj}_L"] \arrow[d, mapsto, "S^{-1}"'] & \mathrm{proj}_L(\vec{v}) \arrow[d, mapsto, "S^{-1}"] \\
{[\vec{v}]_{\mathcal{B}}} \arrow[r, mapsto, "B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}"] & {[\mathrm{proj}_L(\vec{v})]_{\mathcal{B}}}
\end{tikzcd}
$$

The matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ <u>represents</u> the linear transformation proj_L in the basis $\mathcal{B}$. The relationship between the two is

$$\boxed{A = SBS^{-1}}$$

<u>Def</u> Two $n \times n$ matrices A and B are <u>similar</u> if there is an invertible matrix S such that

$$A = SBS^{-1}$$

Similar matrices represent the same linear transformation in different bases.

<u>Def</u> Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ be a basis for $\mathbb{R}^m$. The $\mathcal{B}$-<u>matrix</u> of the

linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is the matrix $[T]_{\mathcal{B}}$ such that

$$[T]_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}.$$

If A is the matrix of T, then

$$[T]_{\mathcal{B}} = S^{-1} A S.$$

$$\begin{tikzcd}
\vec{x} \arrow[r, mapsto, "A"] \arrow[d, "S^{-1}"'] & T(\vec{x}) \arrow[d, "S^{-1}"] \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"'] & {[T(\vec{x})]_{\mathcal{B}}}
\end{tikzcd}$$

The columns of $[T]_{\mathcal{B}}$ are the vectors $[T v_i]_{\mathcal{B}}$.

Ex $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

T the linear transformation with matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = S^{-1}AS = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 2 & 6 \end{bmatrix}.$$