Last time

- More coordinates
- Similarity $\qquad [T]_{\mathcal{B}} = S^{-1}AS$
- The $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$ of T

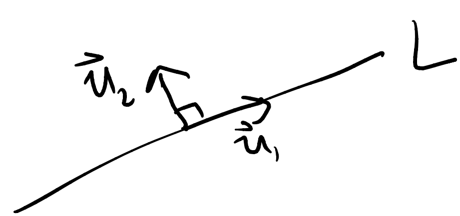
$$
\begin{tikzcd}
\vec{x} \arrow[r, mapsto, "A"] \arrow[d, "S^{-1}"'] & T(\vec{x}) \arrow[d, "S^{-1}"] \\
{[\vec{x}]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"'] & {[T(\vec{x})]_{\mathcal{B}}}
\end{tikzcd}
$$

In the example of projection, $[T]_{\mathcal{B}}$ was particularly simple: it was a <u>diagonal</u> matrix.

The matrix of T in the basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is diagonal precisely when each $T(\vec{v}_i)$ is parallel to $\vec{v}_i$.

Such good coordinates might or might not exist.

We return for now to our original example of coordinates, in which $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \end{bmatrix}$ and



$[\mathrm{proj}_L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. This coordinate system is adapted to the study of projection in that this matrix is diagonal. The same holds for reflection, since $[\mathrm{ref}_L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We'll pursue this idea much further in our study of diagonalization.

Another feature is that $[\vec{x}]_{\mathcal{B}}$ is simple to calculate in terms of dot products. We now highlight this property.

<u>Def</u> The vectors $\vec{u}_1, \ldots, \vec{u}_r$ in $\mathbb{R}^m$ are called <u>orthonormal</u> if

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

In other words, each vector is a unit vector and every two are orthogonal.

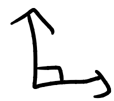
<u>Reminder</u> (1) Two vectors are <u>orthogonal</u> if their dot product is 0.

(2) The **length** of a vector $\vec{v}$ is

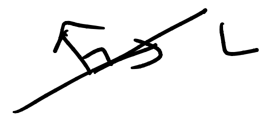
$$|\vec{v}| = \|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + \cdots + v_m^2}.$$

(3) A **unit vector** is a vector of length 1.

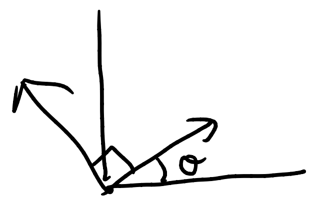
Ex $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$



Ex $\left\{ \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$



Ex $\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$



Ex $\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\boxed{\text{orthogonal vectors are linearly independent}}$

So a collection of linearly independent vectors is automatically a basis for its span, and a collection of m orthonormal vectors in $\mathbb{R}^m$ is automatically a basis for $\mathbb{R}^m$, called an <u>orthonormal basis</u>.

If $\mathcal{B} = \{\vec{u}_1, \ldots, \vec{u}_r\}$ is an orthonormal basis for V, then

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vdots \\ \vec{u}_r \cdot \vec{x} \end{bmatrix}.$$

Using orthonormal bases, we can generalize projection onto a line or plane to orthogonal projection onto any subspace V.

Specifically, given $\vec{x}$ in $\mathbb{R}^m$ and a subspace V, there are unique vectors $\vec{x}^{\perp}$ and $\vec{x}^{\parallel}$ with the following properties

(1) $\vec{x}^{\parallel}$ is in V

(2) $\vec{x}^{\perp}$ is orthogonal to every vector in V

(3) $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$.

The assignment $\vec{x} \longmapsto \vec{x}^{\parallel}$ is a linear transformation called proj_V.