Last time

- Orthonormality
- Orthogonal $\Longrightarrow$ linearly independent
- Toward proj_V

Given $\vec{x}$ in V, we want to find $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$ such that

(1) $\vec{x}^{\parallel}$ is in V

(2) $\vec{x}^{\perp}$ is orthogonal to every vector in V

(3) $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$.

Then we define $\text{proj}_V(\vec{x}) = \vec{x}^{\parallel}$.

To do this, choose an orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_r\}$ for V, and define

$$\vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_r \cdot \vec{x})\vec{u}_r$$

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}.$$

Conditions (1) and (2) are obviously satisfied, and condition (3) is also since given $\vec{v}$ in V, since

$$\begin{aligned}
\vec{u}_i \cdot \vec{x}^{\perp} &= \vec{u}_i \cdot \vec{x} - \vec{u}_i \cdot \vec{x}^{\parallel} \\
&= \vec{u}_i \cdot \vec{x} - (\vec{u}_1 \cdot \vec{x})(\vec{u}_i \cdot \vec{u}_1) + \cdots + (\vec{u}_r \cdot \vec{x})(\vec{u}_i \cdot \vec{u}_r) \\
&= \vec{u}_i \cdot \vec{x} - (\vec{u}_i \cdot \vec{x})(\vec{u}_i \cdot \vec{u}_i)
\end{aligned}$$

$$= \vec{u}_i \cdot \vec{x} - \vec{u}_i \cdot \vec{x}$$
$$= 0,$$

and a general vector in V is a linear combination of the $\vec{u}_i$.

$$\boxed{\mathrm{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_r \cdot \vec{x})\vec{u}_r}$$

This process relied on the existence of an orthonormal basis, which we justify next.

Gram-Schmidt process

(0) Start with a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_r\}$ for the subspace V.

(1) Set $\vec{u}_1 = \dfrac{\vec{v}_1}{|\vec{v}_1|}$.

(2a) Write $\vec{v}_2 = \vec{v}_2^{\,\parallel} + \vec{v}_2^{\,\perp}$ with respect to $\mathrm{Span}(\vec{u}_1)$.

(2b) Set $\vec{u}_2 = \dfrac{\vec{v}_2^{\,\perp}}{|\vec{v}_3^{\,\perp}|}$.

$\vdots$

(ja) Write $\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}$ with respect to $\mathrm{Span}(\vec{u}_1, \dots, \vec{u}_{j-1})$.

(jb) Set $\vec{u}_j = \dfrac{\vec{v}_j^{\perp}}{|\vec{v}_j^{\perp}|}$.

$\vdots$

Then the resulting vectors $\{\vec{u}_1, \dots, \vec{u}_r\}$ are orthonormal by construction, hence an orthonormal basis for V, since $\dim V = r$.

<u>Ex</u> Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. Then

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

(1) $$\vec{u}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(2a) $$\begin{aligned} \vec{v}_2^{\perp} &= \vec{v}_2 - \vec{v}_2^{\parallel} \\ &= \vec{v}_2 - \operatorname{proj}_{\operatorname{span}(\vec{v}_1)}(\vec{v}_2) \\ &= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 \\ &= \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

(2b) $$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{|\vec{v}_2^{\perp}|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(3a) $\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel}$

$$= \vec{v}_3 - \mathrm{proj}_{\mathrm{span}(\vec{u}_1, \vec{u}_2)}(\vec{v}_3)$$

$$= \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

(3b) $\vec{u}_3 = \dfrac{\vec{v}_3^{\perp}}{|\vec{v}_3^{\perp}|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$

The GS process begins with a basis $\{\vec{v}_1, \dots, \vec{v}_r\}$ and returns an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_r\}$ defined recursively by

$$\boxed{\begin{aligned} \vec{u}_j &= \vec{v}_j^{\perp} / |\vec{v}_j^{\perp}| \\ \vec{v}_j^{\perp} &= \vec{v}_j - (\vec{u}_1 \cdot \vec{v}_j)\vec{u}_1 - \cdots - (\vec{u}_{j-1} \cdot \vec{v}_j)\vec{u}_{j-1} \end{aligned}}$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\} \to \boxed{\text{GS}} \to \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$