Last time

- QR factorization
- Orthogonal matrices

Characterizations of orthogonality

(1) The columns of A are an ONB

(2) $A^{-1} = A^T$

(3) $A^T A = I_n$

(4) A preserves length

(5) A preserves dot products (length & angle)

The transpose also permits a clean formula for orthogonal projection.

Given an ONB $\vec{u}_1, \ldots, \vec{u}_r$ for a subspace V, we have

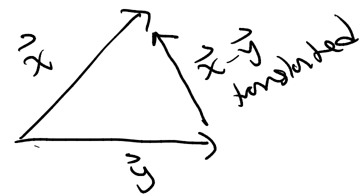
$$\begin{aligned}
\mathrm{proj}_V(\vec{x}) &= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_r \cdot \vec{x})\vec{u}_r \\
&= \vec{u}_1 \vec{u}_1^T \vec{x} + \cdots + \vec{u}_r \vec{u}_r^T \vec{x} \\
&= (\vec{u}_1 \vec{u}_1^T + \cdots + \vec{u}_r \vec{u}_r^T)\vec{x} \\
&= QQ^T \vec{x},
\end{aligned}$$

where $Q = [\vec{u}_1 \cdots \vec{u}_r]$.

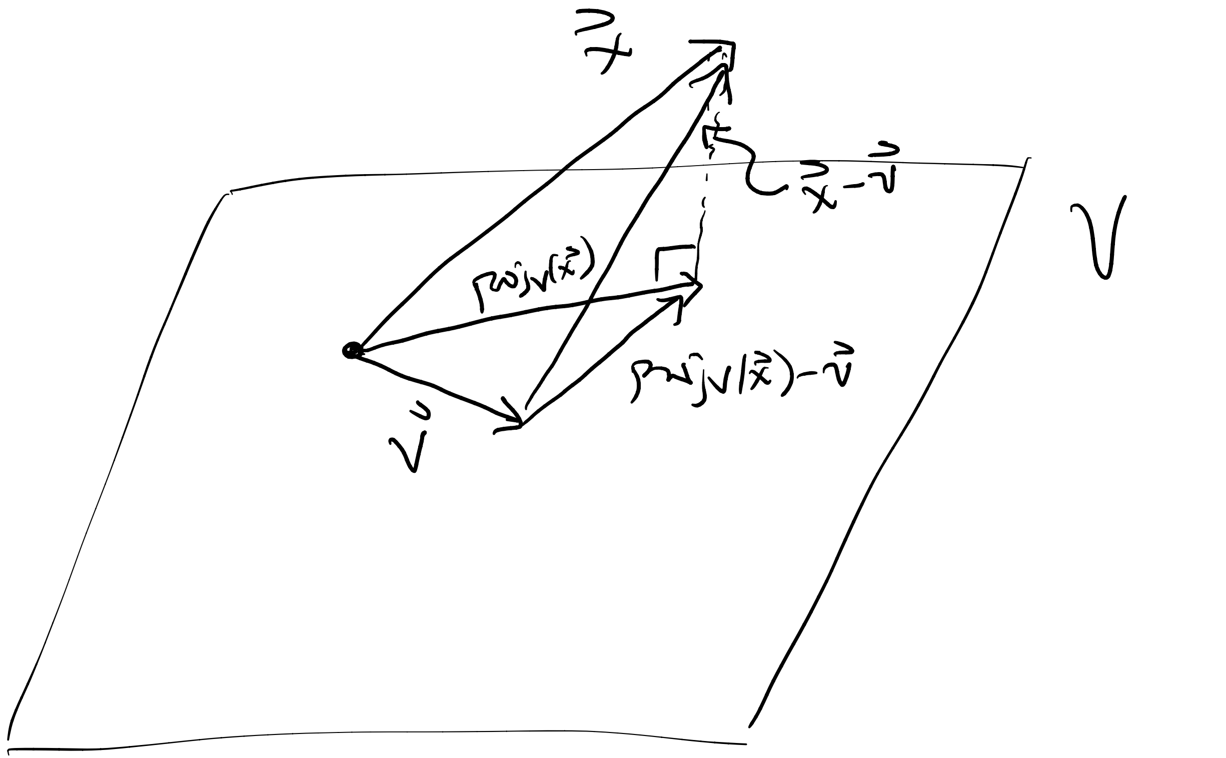
$$\boxed{\text{proj}_V = QQ^T \qquad Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r \\ | & & | \end{bmatrix}}$$

Orthogonal projection is important because it minimizes distance.

Def The distance between two vectors $\vec{x}$ and $\vec{y}$ is the length of their difference $|\vec{x}-\vec{y}|$.



Then $\mathrm{proj}_V(\vec{x})$ is the vector in V <u>closest</u> to $\vec{x}$, i.e., $|\vec{x} - \mathrm{proj}_V(\vec{x})| < |\vec{x} - \vec{v}|$ for every $\vec{v}$ in V different from $\mathrm{proj}_V(\vec{x})$.



By Pythagoras,

$$|\vec{x}-\vec{v}|^2 = |\vec{v}-\text{proj}_V(\vec{x})|^2 + |\vec{x}-\text{proj}_V(\vec{x})|^2$$
$$\leq |\vec{x}-\text{proj}_V(\vec{x})|^2$$

with equality if and only if $|\vec{v}-\text{proj}_V(\vec{x})|^2 = 0$.

Q What to do if $A\vec{x}=\vec{b}$ has no solution?

A Minimize the error $|A\vec{x}-\vec{b}|$.

Def Let A be an $m\times n$ matrix. A vector $\vec{x}^*$ in $\mathbb{R}^n$ is called a least-squares solution to the system $A\vec{x}=\vec{b}$ if

$$|\vec{b}-A\vec{x}^*| \leq |\vec{b}-A\vec{x}|$$

for every vector $\vec{x}$ in $\mathbb{R}^n$.

The collection of vectors $A\vec{x}$ as $\vec{x}$ varies is just the image of A, so we want to minimize the distance from $\vec{b}$ to $\text{im}(A)$. In other words, $\vec{x}^*$ is a least-squares solution precisely when

$$A\vec{x}^* = \text{proj}_{\text{im}(A)}(\vec{b}),$$

i.e., when $\vec{b} - A\vec{x}^*$ is orthogonal to $\text{im}(A)$.