Last time

- Normal equation $A^T A\vec{x} = A^T\vec{b}$
- Problem: deciding invertibility
- $\det(A) = \vec{u}\cdot(\vec{v}\times\vec{w})$, $A = [\vec{u}\ \vec{v}\ \vec{w}]$

Recall $\vec{v}\times\vec{w} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$

(1) $\vec{v}\times\vec{w} = \vec{0}$ if $\vec{v}$ and $\vec{w}$ are parallel

(2) $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$

$\det(A) = 0 \iff \vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$

$\iff$ either $\vec{v}$ and $\vec{w}$ are parallel or $\vec{u}$ lives in the plane spanned by $\vec{v}$ and $\vec{w}$

$\iff \{\vec{u}, \vec{v}, \vec{w}\}$ is not linearly independent

$\iff$ A is not invertible.

If we unpack this formula, we get the following mnemonic, called Sarrus' rule:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$- \quad - \quad - \qquad\qquad + \quad + \quad +$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}}_{A} \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}$$

$$\det(A) = 50 + 84 + 96 - 105 - 48 - 80$$
$$= 230 - 233 = -3$$

Invertible!

We might hope to add up products of entries (with signs) in a general $n \times n$ matrix to define $\det(A)$ in general. How to decide which products and signs?

In the 2×2 case, we have the terms

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ \qquad\qquad -$$

For 3x3, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \qquad\qquad +$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ \qquad\qquad -$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$- \qquad\qquad -$$

__Def__ A __pattern__ in an $n \times n$ matrix A is a choice of n entries of A such that there is exactly one entry in each row and column. Two entries in a pattern P are __inverted__ if one is above and to the right of the other. We say P is __odd__ or __even__ according to whether it has an odd or even number of inversions.

The <u>determinant</u> of A is

$$\det(A) = \sum \operatorname{sgn} P \cdot \operatorname{prod} P,$$

where the summation is over all patterns P, $\operatorname{prod} P$ is the product of the entries of P, and

$$\operatorname{sgn} P = \begin{cases} +1 & \text{if } P \text{ is even} \\ -1 & \text{if } P \text{ is odd.} \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 0$$

$+$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 2$$

$+$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 2$$

$+$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 3$$

$-$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 1$$

$-$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 1$$

$-$

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad 7$$

$$\det(A) = -3 \cdot 2 \cdot 1 \cdot 8 \cdot 5 \cdot 2 = -480$$

Goal $\det(A) = 0 \iff A$ is not invertible

Idea We test invertibility with row reduction, so let's try to relate determinants and row operations.

For 2×2 matrices, this is easily done.

Matrix	Determinant
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$ad-bc$
$\begin{bmatrix} c & d \\ a & b \end{bmatrix}$	$cb-da = -(ad-bc)$
$\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$	$kad-kbc = k(ad-bc)$
$\begin{bmatrix} a & b \\ c+ka & d+kb \end{bmatrix}$	$ad+kab-bc-kab = ad-bc$