Last time

- Affine simplices
- Barycentric subdivision $\quad S : L_*(K) \to L_*(K)$
$$S \simeq \mathrm{id}$$

Def The <u>barycentric subdivision</u> of a singular n-simplex $\sigma : \Delta^n \to X$ is the chain
$$S\sigma = \sigma_\#(S\,\mathrm{id}_{\Delta^n})$$

Lemma 3 $S : C_*(X) \to C_*(X)$ is a chain map chain homotopic to $\mathrm{id}_{C_*(X)}$.

Proof We calculate that

$$
\begin{aligned}
\partial S\sigma &= \partial \sigma_{\#}(S\,\mathrm{id}_{\Delta^n}) \\
&= \sigma_{\#}(S\partial\,\mathrm{id}_{\Delta^n}) \\
&= \sigma_{\#} S\left(\sum_{i=0}^{n}(-1)^i \eta_i\right) \\
&= \sum_{i=0}^{n}(-1)^i \sigma_{\#} S \eta_i \\
&= \sum_{i=0}^{n}(-1)^i \sigma_{\#}(\eta_i)_{\#} S\,\mathrm{id}_{\Delta^{n-1}} \\
&= \sum_{i=0}^{n}(-1)^i (\sigma\circ\eta_i)_{\#} S\,\mathrm{id}_{\Delta^{n-1}} \\
&= \sum_{i=0}^{n}(-1)^i S(\sigma\circ\eta_i) \\
&= S\partial\sigma.
\end{aligned}
$$

For the second, define $T: C_*(X) \to C_*(X)$ by

$$T\sigma = \sigma_\# T(\mathrm{id}_{\Delta^n}),$$

and a similar calculation shows that

$\partial T + T\partial = S - \mathrm{id}$. $\square$

<u>Exercise</u> If Γ_1 is a chain homotopy from φ_1 to φ_2 and Γ_2 from ψ_1 to ψ_2, then $\psi_1 \circ \Gamma_1 + \Gamma_2 \circ \varphi_2$ is a chain homotopy from $\psi_1 \circ \varphi_1$ to $\psi_2 \circ \varphi_2$.

Corollary For $r \geq 0$, $S^r : C_*(X) \to C_*(X)$ is chain homotopic to $\mathrm{id}_{C_*(X)}$ via the chain homotopy

$$T_r := \sum_{0 \leq i < r} T S^i .$$

Lemma 4 For $v, w \in \mathrm{im}([v_0, \ldots, v_m])$, we have
$$|v - w| \leq \max |v - v_j| .$$
In particular, $\mathrm{diam}\, \mathrm{im}([v_0, \ldots, v_m]) \leq \max |v_i - v_j|$.

Proof Writing $w = \sum_j t_j v_j$, we have

$$|v-w| = \left| \sum_j t_j v - \sum_j t_j v_j \right|$$
$$\leq \sum_j t_j |v - v_j|$$
$$\leq \left(\sum_j t_j\right) \max |v - v_j|.$$

$\square$

<u>Lemma 5</u> Every face of $S[v_0, \dots, v_m]$ has diameter at most $\frac{m}{m+1} \max |v_i - v_j|$.

<u>Proof</u> Writing $[w_0, \dots, w_m]$ for such a face, it suffices to show by Lemma 4 that $|w_k - w_\ell| \leq \frac{m}{m+1} |v_i - v_j|$ for every $0 \leq k, \ell \leq m$.

WLOG and by induction, $w_k = \frac{1}{m+1}\sum_{i=0}^{m} v_i$, the barycenter. By Lemma 4,

$$
\begin{aligned}
|w_k - w_\ell| &\le \max_j |w_k - v_j| \\
&= \max_j \left| \frac{1}{m+1}\sum_{i=0}^{m} v_i - v_j \right| \\
&= \max_j \left| \frac{1}{m+1}\sum_{i\neq j} v_i - \frac{m}{m+1} v_j \right| \\
&\le \frac{m}{m+1} \max |v_i - v_j| .
\end{aligned}
$$

$\square$

Lebesgue number lemma Given an open cover $\{V_\alpha\}_{\alpha \in A}$ of $K \subseteq \mathbb{R}^n$ with K compact, there exists $\delta > 0$ such that every subset of K of diameter less than δ is contained in some V_α.

Proof of subdivision thm For every $\sigma : \Delta^n \to X$, Lemma 5 and the Lebesgue number lemma (applied to $\{\sigma^{-1} \mathcal{U}_\alpha\}_{\alpha \in A}$) imply that $S^r \sigma \in C_*^{\mathcal{U}}(X)$ for r sufficiently large. Write $r(\sigma)$ for the least such r, and define $\Gamma : C_*(X) \to C_*(X)$ by

$$\Gamma\sigma = T_{r(\sigma)}\sigma.$$

We claim that $\rho(\sigma) := \sigma - \partial\Gamma\sigma - \Gamma\partial\sigma$ lies in $C_*^{\theta}(X)$; assuming so, it follows that $\rho : C_*(X) \to C_*^{\theta}(X)$ is a chain map and Γ a chain homotopy between $\iota\rho$ and $\mathrm{id}_{C_*(X)}$. But $\rho(\iota(\sigma)) = \sigma$ for $\sigma \in C_*^{\theta}(X)$, since $r(\sigma) = 0$ in this case, so the theorem follows. For the claim, we have

$$\sigma - \partial\Gamma\sigma - \Gamma\partial\sigma = \sigma - \partial T_{r(\sigma)}\sigma - \Gamma\partial\sigma$$

$$= \sigma - \left(-T_{r(\sigma)}\partial\sigma + \sigma - S^{r(\sigma)}\sigma\right) - \Gamma\partial\sigma$$

$$= S^{r(\sigma)}\sigma + \left(T_{r(\sigma)} - \Gamma\right)\partial\sigma$$

$$= S^{r(\sigma)}\sigma + \sum_j (-1)^j \left(T_{r(\sigma)} - T_{r(\sigma\circ\eta_j)}\right)(\sigma\circ\eta_j)$$

$$= S^{r(\sigma)}\sigma + \sum_j (-1)^j \sum_{i=r(\sigma\circ\eta_j)}^{r(\sigma)-1} TS^i(\sigma\circ\eta_j),$$

since $r(\sigma) \geq r(\sigma\circ\eta_j)$. But S and T preserve $C_*^{\mathcal{O}}(X)$ (exercise), so each term lives in $C_*^{\mathcal{O}}(X)$, as desired. $\square$