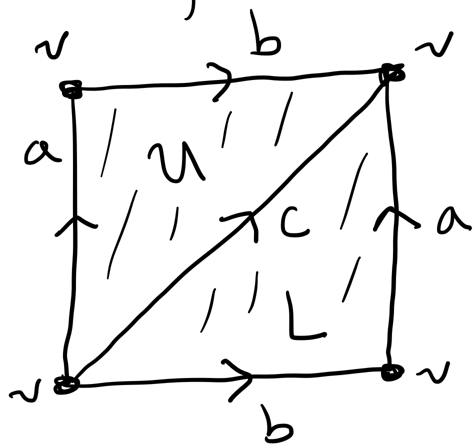
Last time

- Proof of subdivision

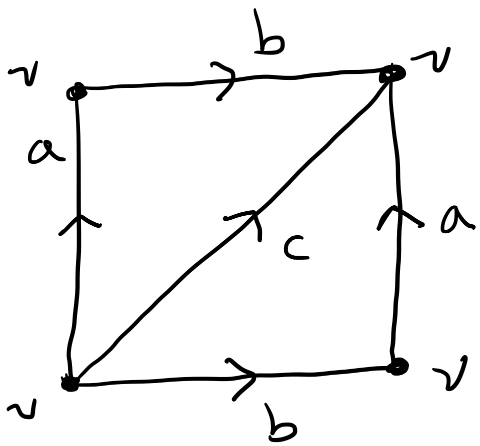
Our debt is paid! Yet we have very few calculations for our efforts.

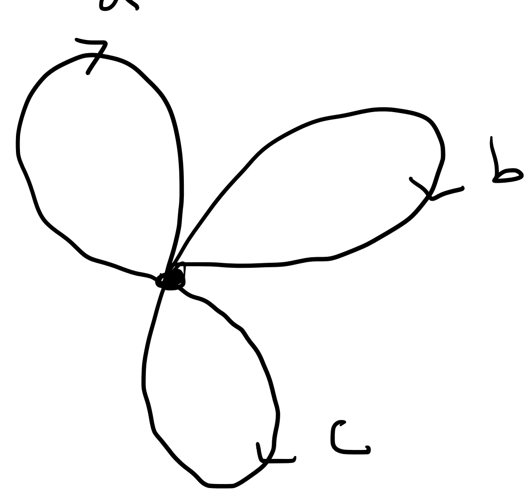
<u>Problem</u> Calculate $H_*(T^2)$, $T^2 = S^1 \times S^1$.

We aim to use the long exact sequence in relative homology for the subspace A given by the union of the edges in our standard picture of the torus:

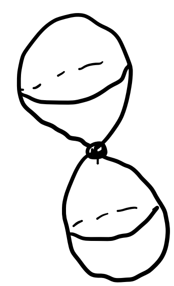
$T^2 =$ 

$=$ 

$A =$ 

$=$ 

$\cong S^1 \vee S^1 \vee S^1$

$$T^2/A = U/\partial U \coprod_{\{v\}} L/\partial L =$$  $\cong S^2 \vee S^2$

<u>Exercise</u> (T^2, A) is a "good pair" (A is a deformation retract of an open neighborhood).

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(T^2) \to \tilde{H}_n(T^2/A) \to \tilde{H}_{n-1}(A) \to \cdots$$

<u>Prop</u> Let (X, x_0) and (Y, y_0) be spaces with base-points. If both are good pairs, then

$$\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

<u>Proof</u> If $x_0 \in U$ and $y_0 \in V$ are deformation retracts, we have the Mayer-Vietoris sequence

$$\cancel{\tilde{H}_n(U \vee V)}^{\,0} \to \tilde{H}_n(X \vee V) \oplus \tilde{H}_n(Y \vee U) \to \tilde{H}_n(X \vee Y) \to \cancel{\tilde{H}_{n-1}(U \vee V)}^{\,0}$$

$$\cong \quad \tilde{H}_n(X) \oplus \tilde{H}_n(Y).$$

$\square$

So, in our case,

$$\tilde{H}_n(A) \cong \tilde{H}_n(S^1 \vee S^1 \vee S^1) \cong \begin{cases} \mathbb{Z}^3 & n=1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{H}_n(X/A) \cong \tilde{H}_n(S^2 \vee S^2) \cong \begin{cases} \mathbb{Z}^2 & n=2 \\ 0 & \text{otherwise.} \end{cases}$$

The LES becomes the sequences

$$0 \to H_n(T^2) \to 0, \quad n>2$$

$$0 \to H_2(T^2) \to \mathbb{Z}^2 \xrightarrow{\delta} \mathbb{Z}^3 \to H_1(T^2) \to 0$$

$$\tilde{H}_n(T^2) \cong \begin{cases} \ker \delta & n=2 \\ \mathbb{Z}^3/\operatorname{im}\delta & n=1 \\ 0 & \text{otherwise.} \end{cases}$$

We calculate δ componentwise using naturality: the map of pairs $(u, \partial u) \subseteq (T^2, A)$ induces a commuting diagram

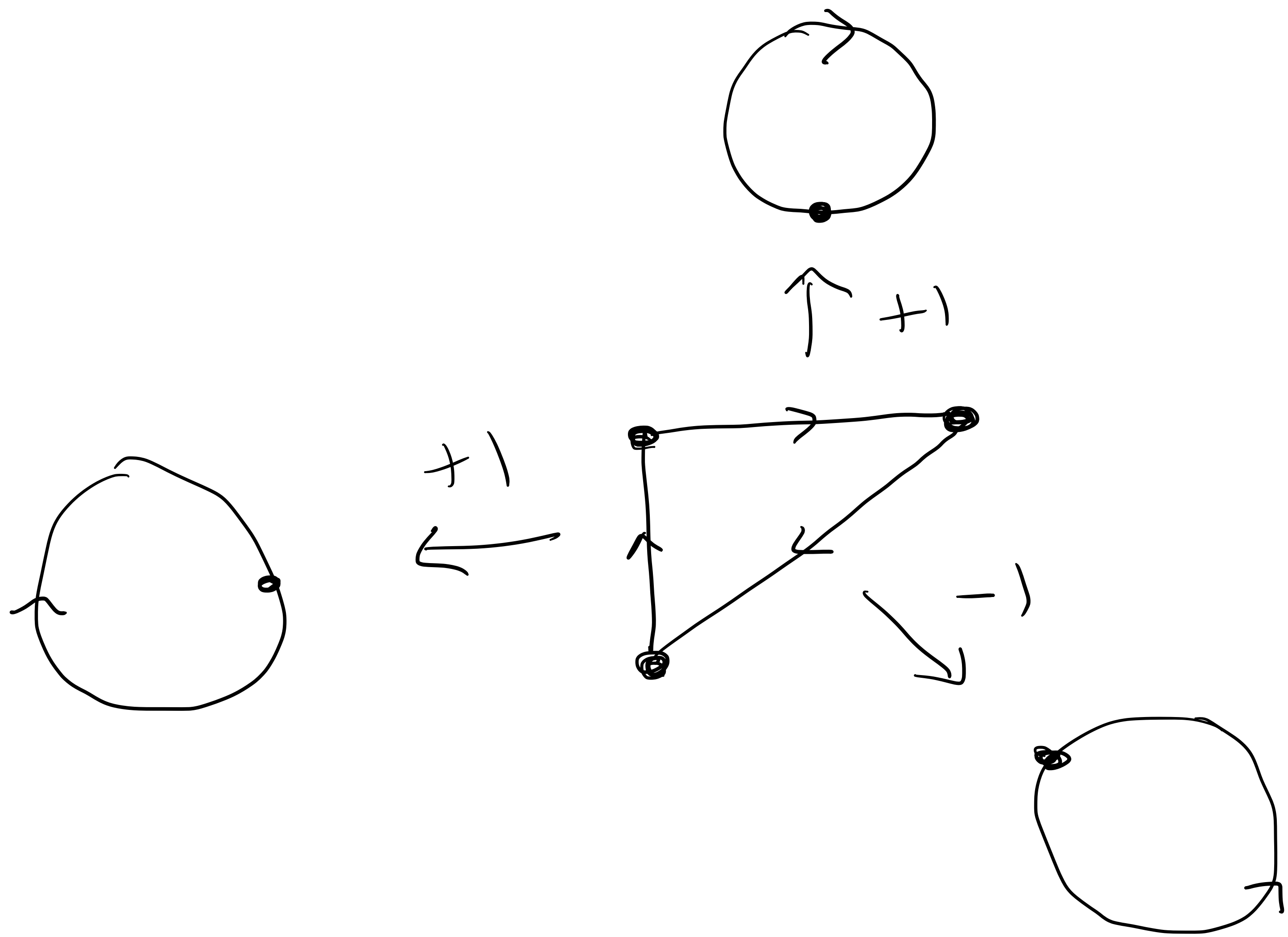
$$\begin{tikzcd}
(1,0) & \mathbb{Z}^2 \cong H_2(T^2, A) \arrow[r, "\delta"] & H_1(A) \cong H_1(S^1 \vee S^1 \vee S^1) & (\deg(f_1), \deg(f_2), \deg(f_3)) \\
1 \arrow[u, mapsto] & \mathbb{Z} \cong H_2(\Delta^2, \partial\Delta^2) \arrow[u, "u"] \arrow[r, "\delta"] & H_1(\partial\Delta^2) \cong H_1(S^1) \arrow[u] & 1 \arrow[u, mapsto]
\end{tikzcd}$$

$$1 \longmapsto 1$$

where f_i is the composite

$$S^1 \longrightarrow S^1 \vee S^1 \vee S^1 \xrightarrow{\text{crush all but } i^{\text{th}} \text{ factor}} S^1 .$$

In our example, these are the maps



$$\delta(1,0) = (1,1,-1)$$

$\Longrightarrow$

$$\delta(0,1) = (1,1,-1)$$

<u>Takeaway</u>

$$\ker(\delta) = \langle (1,-1) \rangle$$

$$\operatorname{Im}(\delta) = \langle (1,1,-1) \rangle$$

$$H_2(T^2) \cong \mathbb{Z}$$

$\Longrightarrow$

$$H_1(T^2) \cong \mathbb{Z}^3 / \langle (1,1,-1) \rangle \cong \mathbb{Z}^2$$

__Observation__ This calculation is identical to the motivational "computations" we did in Lecture 1 (and to the simplicial homology from the homework).

$$
\begin{tikzcd}
\text{small combinatorial description} \arrow[r, Rightarrow] & \text{small chain complex} \arrow[r, Rightarrow] & \text{Homology} \arrow[d, equal, "?\ !"] \\
\text{large combinatorial description} \arrow[r, Rightarrow] & \text{large chain complex} \arrow[r, Rightarrow] \arrow[u, Rightarrow, "\text{LES}"'] & \text{Homology}
\end{tikzcd}
$$

Observation What made this calculation work was that we built T^2 by gluing disks to circles along their boundaries.

Observation When we do this, the boundary map can be calculated using degrees.

The spaces that can be built inductively by attaching disks of increasing dimension along their boundaries are called CW complexes.

<u>Def</u> Given a subspace $A \subseteq X$, we say that X is obtained from A by <u>attaching cells</u> of dimension n if there is a set of maps $\{e_i : (D^n, S^{n-1}) \longrightarrow (X, A)\}_{i \in I}$ such that

$$X \cong A \amalg \coprod_{i \in I} D^n \Big/ \begin{matrix} e_i(x) \sim x, \\ x \in S^{n-1} \end{matrix}$$

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r, hook] \arrow[d] & \coprod D^n \arrow[d] \\ A \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

<u>Rmk</u> (1) The complement $X \setminus A$ is homeomorphic to $\coprod_{i \in I} \mathring{D}^n$, each component of which is called an <u>open n-cell</u>.

(2) A <u>closed n-cell</u> is the closure of an open n-cell, which may not be homeomorphic to S^{n-1}!

(3) The map e_i is a characteristic map for its cell. A cell has many characteristic maps leading to homeomorphic spaces.

(4) If $e: D^n \to X$ is a characteristic map for a cell, then $e|_{\mathring{D}^n}$ is a homeomorphism onto its image, and $e(\mathring{D}^n)$ is open in X.

Ex T^2 is obtained from $S^1 \vee S^1$ by attaching a cell of dimension 2.