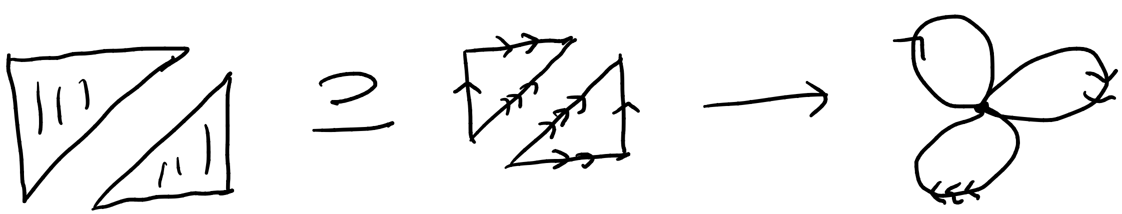
Last time

- $H_*(T^2)$
- Attaching cells

Ex T^2 is obtained from $S^1 \vee S^1 \vee S^1$ by attaching two cells of dimension 2:



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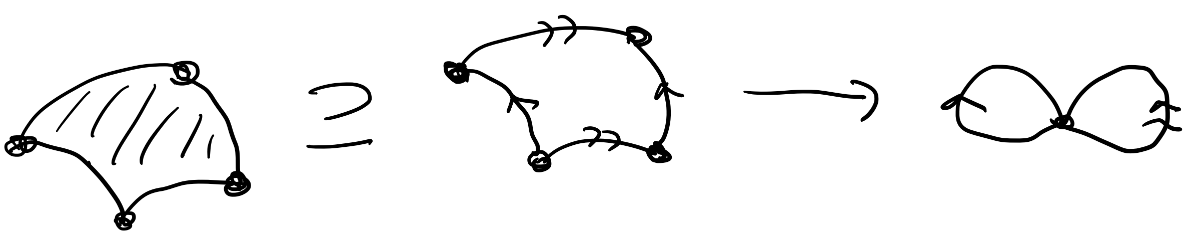


<u>Def</u> A <u>CW complex</u> structure on a space X is a collection of subspaces $\{X_n\}_{n\geq 0}$ such that

(1) $X = \bigcup_{n\geq 0} X_n$

(2) X_n is obtained from X_{n-1} by attaching cells of dimension n ($X_{-1} = \emptyset$ by convention).

(3) $U \subseteq X$ is open (resp. closed) iff $U \cap X_n$ is open (resp. closed) in X_n for every n.

<u>Rmks</u> (1) X_0 is discrete

(2) X_n is closed in X

(3) closed cells are closed in X, and

open cells are open in X_n (but usually not in X).

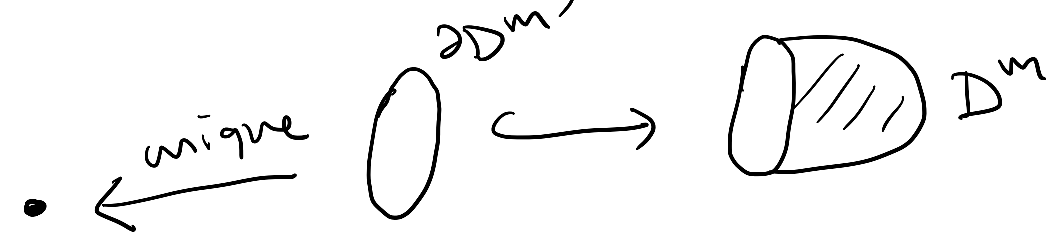
(4) The open cells form a partition of X.

(5) X_n is the <u>n-skeleton</u> of X. It is the union of all (closed or open) cells of dimension at most n.

(6) If $X = X_n$ for some n, then X is <u>finite dimensional</u>.

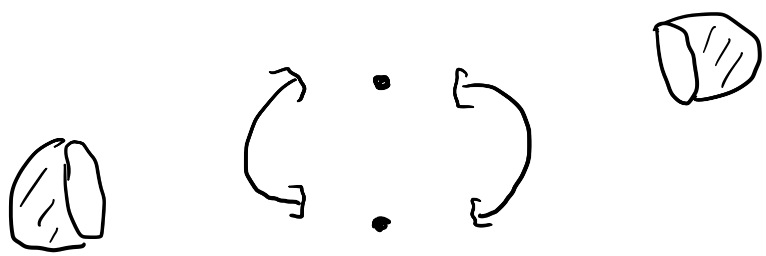
<u>Example</u> A discrete space is uniquely a CW complex with $X_0 = X_1 = \cdots = X$.

<u>Example</u> S^m has a CW complex structure with one 0-cell, one m-cell, and no other cells:



Rmk This example illustrates the utility of CW complexes: the flexibility of allowing arbitrary attaching maps allows for a very small number of cells.

Example S^m has another CW complex structure with two n-cells for every $0 \le n \le m$:



This CW structure has more cells but is suitable for inductive study.

Rmk These examples show that CW structures are highly non-unique.

Example Any Δ-complex is a CW complex with one n-cell for every n-simplex.

The homological import of CW complexes stems from the following result.

Prop Let X be a CW complex with set of k-cells I_k.

$$(1) \quad H_n(X_k, X_{k-1}) \cong \begin{cases} \mathbb{Z}\langle I_k \rangle, & n=k \\ 0 & \text{otherwise.} \end{cases}$$

(2) $X_k \subseteq X_{k+1}$ induces an isomorphism on H_n for $n \notin \{k, k+1\}$, a surjection on H_k and an injection on H_{k+1}.

(3) $H_n(X_k) = 0$ for $k < n$.

(4) $X_k \subseteq X$ induces an isomorphism on H_n for $n < k$ and a surjection on H_k.

<u>Lemma 1</u> If X is a CW complex, then (X_n, X_{n-1}) is a good pair for every $n \geq 0$.

<u>Lemma 2</u> If K is a compact subspace of a CW complex X, then K intersects only finitely many cells of X.

Proof of proposition (1) By Lemma 1, it suffices to calculate the reduced homology of the quotient

$$X_k / X_{k-1} \cong X_{k-1} \cup_\partial \Big(\coprod_{I_k} D^k \Big) \Big/ X_{k-1}$$

$$\cong \coprod_{I_k} D^k \Big/ \coprod_{I_k} \partial D^k$$

$$\cong \bigvee_{I_k} S^k ,$$

the homology of which is as claimed.

(2) We have the exact sequence

$$H_{n+1}(X_{k+1}, X_k) \to H_n(X_k) \to H_n(X_{k+1}) \to H_n(X_{k+1}, X_k),$$

so (1) implies the claim.

(3) By (2), $X_0 \subseteq X_k$ induces a surjection on H_n for $n \geq k$, and $H_n(X_0) = 0$ for $n > 0$, since X_0 is discrete.

(4) By (2), $X_k \subseteq X_\ell$ induces an isomorphism on H_n for $n < k$ and a surjection on H_k for any $k \leq \ell$. If X is finite dimensional, the claim follows; in general, Lemma 2 implies that the image of every singular simplex of X factors through some skeleton. In particular, an n-cycle

with $n \le k$ arises from X_ℓ for some $\ell \ge k$, hence from X_k, implying surjectivity. For injectivity, apply the same reasoning to a bounding chain as in the lemma on exhaustions by opens.

$\square$