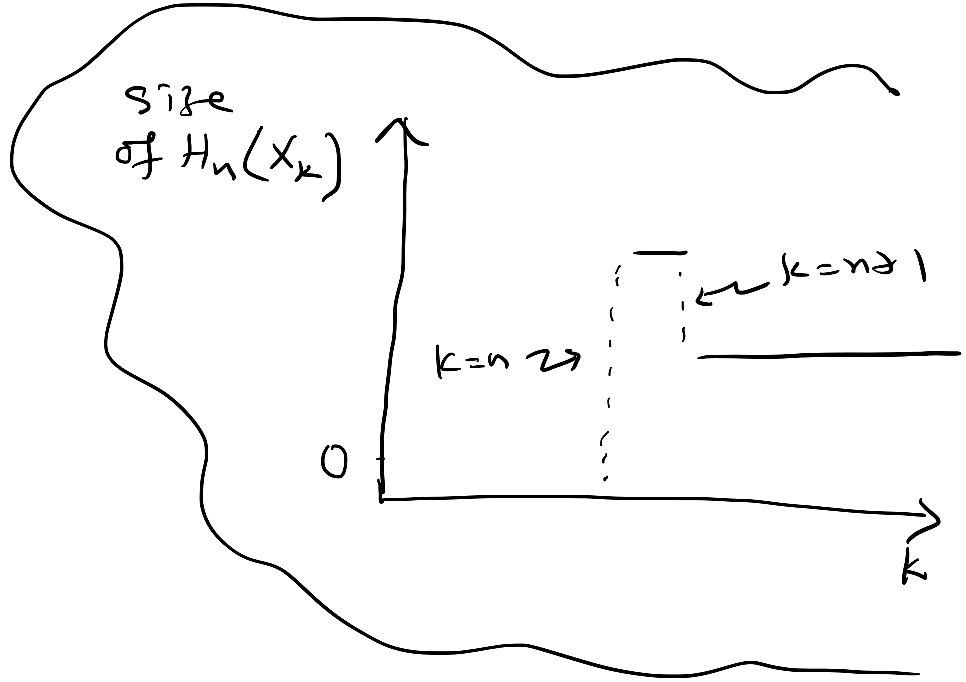
Last time

- CW complexes
- Homological properties



<u>Def</u> Let X be a CW complex. The group of <u>cellular n-chains</u> of X is

$$C_n^{CW}(X) := H_n(X_n, X_{n-1}).$$

These groups form a chain complex with differentials the composites in the diagram

$$
\begin{tikzcd}
\cdots \arrow[dr, dashed] & C_n^{cw}(X) \arrow[rrrr, dashed, "d_n^{cw}"] \arrow[d, equal] & & & & C_{n-1}^{cw}(X) \arrow[d, equal] & \\
& H_n(X_n, X_{n-1}) \arrow[dr, "\delta"] & & & & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, dashed] & \cdots \\
& & H_{n-1}(X_{n-1}) \arrow[urrr] \arrow[dr] & & & & \\
& H_{n-1}(X_{n-2}) \arrow[ur] & & H_{n-1}(X_n) \arrow[dr, dashed] & & & \\
\cdots \arrow[ur, dashed] & & & & \cdots & &
\end{tikzcd}
$$

Notice that $d_{n-1}^{cw} \circ d_n^{cw}$ is the composite

$$
\begin{tikzcd}
H_n(X_n, X_{n-1}) \arrow[d] & & H_{n-2}(X_{n-2}, X_{n-3}) \\
H_{n-1}(X_{n-1}) \arrow[r] \arrow[rr, bend left, "0"] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r] & H_{n-2}(X_{n-2}) \arrow[u]
\end{tikzcd}
$$

.

Def The cellular homology of X, denoted $H_*^{CW}(X)$ is the homology of $C_*^{CW}(X)$.

Thm There is a natural isomorphism

$$H_*(X) \cong H_*^{CW}(X).$$

Rmk The nature of the naturality will be clarified in time.

Proof By definition $H_n^{CW}(X) = \ker d_n^{CW} / \operatorname{im} d_{n+1}^{CW}$.

Consider the following commuting diagram

$$\begin{tikzcd}
H_{n+1}(X_{n+1}, X_n) \arrow[rr, "d_{n+1}^{CW}"] \arrow[dr, "\delta"'] & & H_n(X_n, X_{n-1}) \arrow[rr, "d_n^{CW}"] \arrow[dr] & & H_{n-1}(X_{n-1}, X_{n-2}) \\
& H_n(X_n) \arrow[ur, "(\star)"] & & H_{n-1}(X_{n-1}) \arrow[ur] & \\
0 & & 0 & & \\
\cancel{H_n(X_{n-1})} \arrow[u] \arrow[uur] & & \cancel{H_{n-1}(X_{n-2})} \arrow[u] \arrow[uur] & &
\end{tikzcd}$$

By exactness, the hooked arrows are injective, so the starred arrow induces an isomorphism

$$H_n(X_n)/\operatorname{im}\delta \cong \ker d_n^{CW}/\operatorname{im} d_{n+1}^{CW} = H_n^{CW}(X).$$

But from the exact sequence

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\delta} H_n(X_n) \to H_n(X_{n+1}) \to \cancelto{0}{H_n(X_{n+1}, X_n)},$$

we have $H_n(X_n)/\operatorname{im}\delta \cong H_n(X_{n+1})$ and the latter is isomorphic to $H_n(X)$ by the proposition. $\square$

We return to prove the lemmas.

Def A *subcomplex* of a CW complex is a closed subspace that is a union of open cells.

Example $X_n \subseteq X$ is a subcomplex.

Exercise A subcomplex $A \subseteq X$ is a CW complex with $A_n = A \cap X_n$.

<u>Prop</u> The following statements are equivalent and hold for a CW complex X.

(1) A subspace $A \subseteq X$ intersecting every open cell in at most one point is closed and discrete.

(2) A compact subspace $K \subseteq X$ is contained in a finite union of open cells.

(3) A compact subspace $K \subseteq X$ is contained in a finite subcomplex.

<u>Proof</u> (1) $\Rightarrow$ (2) Given $K \subseteq X$ compact, let $A \subseteq K$ consist of one point arbitrarily chosen from each open cell intersecting K. Then

A is compact and discrete by (1), hence finite.

(2) $\Rightarrow$ (3) By (2), it suffices to show that a finite union of closed cells is contained in a finite subcomplex. Since (exercise) a finite union of finite subcomplexes is a finite subcomplex, we may take K itself to be a closed cell with characteristic map $e: D^n \to X$. By induction, $e(\partial D^n)$ lies in a finite subcomplex $A \subseteq X$, and $A \cup \operatorname{im}(e)$ is a finite subcomplex. For the base case, a 0-cell is itself a finite subcomplex.

(3) $\Rightarrow$ (1) Suppose that A intersects each

open cell in at most one point. By (3), the intersection of A with a closed cell e is contained in a finite subcomplex, hence $A \cap e$ is finite by our assumption on A, hence closed (exercise: points are closed in CW complexes). Fixing $x \in A$, $A \setminus \{x\}$ satisfies the same hypothesis, so $A \setminus \{x\} \cap e$ is also closed, so $A \cap e$ is discrete. Since the intersection of A with every closed cell is closed and discrete, so is A (exercise).

To prove that all three hold, we first observe that (1) holds for X_0. Supposing

(1) (hence (2) and (3)) hold for X_n, note that the proof of (2) $\Rightarrow$ (3) for X_{n+1} only uses (2) for X_n; therefore, (3) holds for X_{n+1}, hence (1) as well. By induction, all three hold for every skeleton of X. For the same reason as before, (3) follows for X, hence also (1) and (2). $\square$